# CSci 8980: Advanced Topics in Graphical Models

## Dirichlet Processes

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- One can define a *measure* $\mu$ on a measurable space

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- Parametric vs non-parametric Bayes

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- The joint distribution over $(P(A_1), \ldots, P(A_m))$

$$P(A_i) = \sum_{\substack{(\gamma_1,\cdots,\gamma_m) \\ \gamma_i = 1}} P(B_{\gamma_1,\cdots,\gamma_m})$$

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- Then the distribution of $(P(B_1), \cdots, P(B_k))$ is identical to that of

$$\left( \sum_1^{r_1} P(B'_i), \sum_{r_1+1}^{r_2} P(B'_i), \cdots, \sum_{r_{k-1}+1}^{k'} P(B'_i) \right)$$

## A Key Lemma

- Lemma: If the joint distribution $(P(B_1), \cdots, P(B_k))$ satisfies the consistency condition, and, if for arbitrary sets $(A_1, \ldots, A_m)$, the joint distribution is constructed as outlined earlier, then there exists $\mathcal{P}$ which yields these distribution.

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- Can inference be tractably done over such models?

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- Key Property: If $(X_1,\ldots,X_k) \sim D(\alpha_1,\ldots,\alpha_k)$, and $r_1,\ldots,r_\ell$ are integers such that $0 < r_1 < \cdots < r_\ell$ then

$$\left(\sum_{1}^{r_1} X_i, \sum_{r_1+1}^{r_2} X_i, \cdots, \sum_{r_{\ell-1}+1}^{k} X_i\right) \sim D\left(\sum_{1}^{r_1} \alpha_i, \sum_{r_1+1}^{r_2} \alpha_i, \cdots, \sum_{r_\ell+1}^{k} \alpha_i\right)$$

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- In particular, the marginal distribution of $X_j \sim B(\alpha_j, \sum_1^k \alpha_i - \alpha_j)$ where

$$B(\alpha,\beta) = f(x|\alpha,\beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$$

# Gamma and Dirichlet

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- Key property: If $X_i \sim \Gamma(\alpha_i, \theta), i = 1, \ldots, k$, then

$$\sum_{i=1}^{k} X_i \sim \Gamma\left(\sum_{i=1}^{k} \alpha_i, \theta\right)$$

# Gamma and Dirichlet

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$$\Gamma(\alpha, \theta) = f(x|\alpha, \theta) = \frac{\exp(-x/\theta)}{\theta^\alpha \Gamma(\alpha)} x^{\alpha-1}$$

- Key property: If $X_i \sim \Gamma(\alpha_i, \theta), i = 1, \ldots, k$, then

$$\sum_{i=1}^{k} X_i \sim \Gamma\left(\sum_{i=1}^{k} \alpha_i, \theta\right)$$

- Let $Z_i = \frac{X_i}{\sum_{i=1} X_i}$, then

$$(Z_1, \ldots, Z_k) \sim D(\alpha_1, \ldots, \alpha_k)$$

# Gamma, Exponential, Geometric

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- Discrete version of exponential is the geometric distribution

$$f(k|q) = (1-q)^{k-1} q$$

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  - Similarly for each $X_i$
- If prior distribution is $D(\alpha_1, \cdots, \alpha_k)$, then posterior

$$P(X_1, \ldots, X_k | X = j) = D(\alpha_1^{(j)}, \cdots, \alpha_k^{(j)})$$

where

$$\alpha_i^{(j)} = \begin{cases} \alpha_i & \text{if } i \neq j \\ \alpha_j + 1 & \text{if } i = j \end{cases}$$

# Dirichlet Processes

- Definition: Let $\alpha$ be a non-negative finite measure on $(X, \mathcal{A})$. Then $P$ is a Dirichlet Process on $(X, \mathcal{A})$ with parameter $\alpha$ if for every $k = 1, 2, \cdots$, and a partition $(B_1, \cdots, B_k)$ of $X$, the distribution of $(P(B_1), \cdots, P(B_k))$ is Dirichlet $D(\alpha(B_1), \cdots, \alpha(B_2))$.

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- For any $A \in \mathcal{A}$, $E[P(A)] = \frac{\alpha(A)}{\alpha(X)}$
- Let $Q$ be a fixed probability measure on $(X, A)$ with $Q \ll \alpha$. Then for any $m$, and any $A_1, \ldots, A_m$, and $\epsilon > 0$,

$$\mathcal{P}\{|P(A_i) - Q(A_i)| < \epsilon, i = 1, \ldots, m\} > 0$$

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- Prop 3: The posterior distribution given $X$ is the DP with parameter $\alpha + \delta_X$
  - Posterior given $X_1, \ldots, X_n$ is the DP with parameter $\alpha + \sum_{i=1}^{n} \delta_{X_i}$

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  - $Q(I = n | (\pi, Y)) = p_n = \pi_n \prod_{1 \le m \le (n-1)} (1 - \pi_m)$ so that

$$\sum_{1 \le m \le n} p_n = 1 - \prod_{1 \le m \le n} (1 - \pi_m) \to 1 \quad w.p.\ 1$$

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- By construction, clearly $P$ can only be discrete

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- In "practice" DPM is a infinite mixture model

# DPM (Contd.)

- Mike Jordan's NIPS'05 Tutorial

# Model-Based Clustering

- A generative approach to clustering:
  - pick one of $K$ clusters from a distribution $\pi = (\pi_1, \pi_2, \ldots \pi_K)$
  - generate a data point from a cluster-specific probability distribution

- This yields a finite mixture model:

$$p(x \mid \phi, \pi) = \sum_{k=1}^{K} \pi_k \, p(x \mid \phi_k),$$

  where $\pi$ and $\phi = (\phi_1, \phi_2, \ldots \phi_K)$ are the parameters, and where we've assumed the same parameterized family for each cluster (for simplicity)

- Data $\{x_i\}_{i=1}^{n}$ are assumed to be generated conditionally IID from this mixture

# Finite Mixture Models (cont)

- Another way to express this: define an underlying measure

$$G = \sum_{k=1}^{K} \pi_k \, \delta_{\phi_k}$$

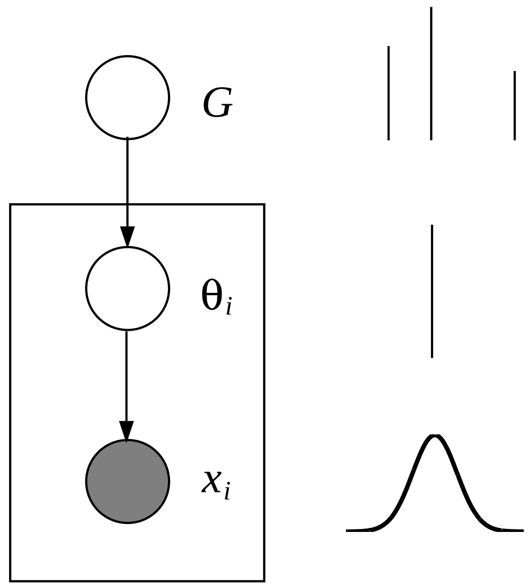
  where $\delta_{\phi_k}$ is an *atom* at $\phi_k$

- And define the process of obtaining a sample from a finite mixture model as follows. For $i = 1, \ldots, n$:

$$\begin{aligned} \theta_i &\sim G \\ x_i &\sim p(\cdot \mid \theta_i) \end{aligned}$$

- Note that each $\theta_i$ is equal to one of the underlying $\phi_k$
  - indeed, the subset of $\{\theta_i\}$ that maps to $\phi_k$ is exactly the $k$th cluster

# Finite Mixture Models (cont)

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# Bayesian Finite Mixture Models

(e.g., Lo; Ferguson; Escobar & West; Robert; Green & Richardson; Neal; Ishawaran & Zarepour)

- Need to place priors on the parameters $\phi$ and $\pi$
- The choice of prior for $\phi$ is model-specific; e.g., we might use conjugate normal/inverse-gamma priors for a Gaussian mixture model
  - let's denote this prior as $G_0$
- Place a symmetric Dirichlet prior, $\mathrm{Dir}(\alpha_0/K, \ldots, \alpha_0/K)$, on the mixing proportions $\pi$
  - the symmetry accords with the (usual) assumption that we could scramble the labels of the mixture components and not change the model
  - the scaling $(\alpha_0/K)$ gives $\alpha_0$ the semantics of a concentration parameter; the prior mean of $\phi_k$ is equal to $1/K$

# Bayesian Finite Mixture Models (cont)

$$
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\phi_k &\sim G_0 \\
\pi_k &\sim \mathrm{Dir}(\alpha_0/K, \ldots, \alpha_0/K) \\
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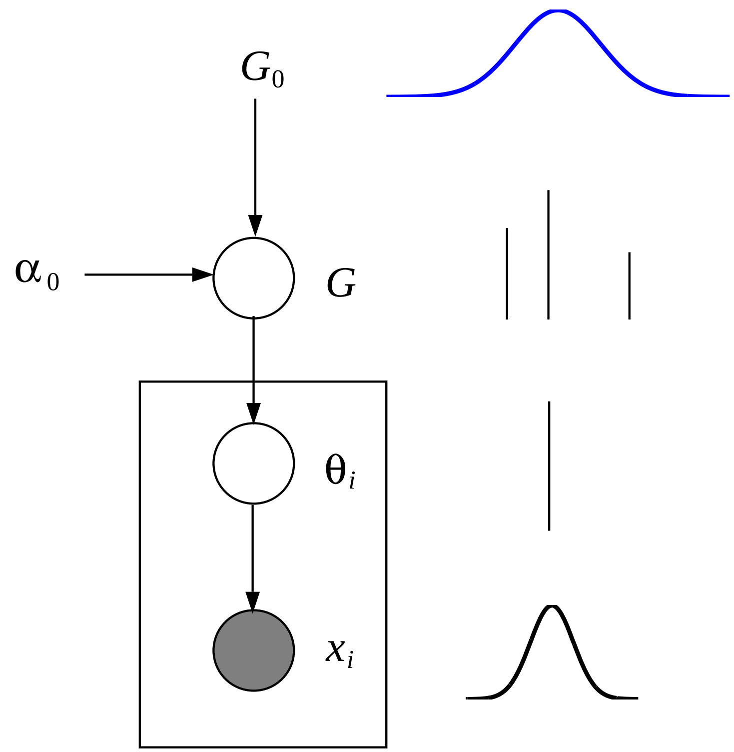



- Note that $G$ is now a *random measure*

# Going Nonparametric—A First Perspective

(e.g., Kingman; Waterson; Patil & Taillie; Liu; Ishwaran & Zarepour)

- Define a countably infinite mixture model by taking $K$ to infinity and hoping that "$G = \sum_{k=1}^{\infty} \pi_k \, \delta_{\phi_k}$" means something, where

$$\begin{aligned} \phi_k &\sim G_0 \\ \pi_k &\sim \mathrm{Dir}(\alpha_0/K, \ldots, \alpha_0/K) \text{ as } K \to \infty \end{aligned}$$

- Several mathematical hurdles to overcome:
  - What is the distribution of any given $\pi_k$ as $K \to \infty$? Does it stabilize at some fixed distribution?
  - Is $\sum_{k=1}^{\infty} \pi_k = 1$ under some suitable notion of convergence?
  - Do we get a few large mixing proportions, or are they all of similar "size"?
  - Do we get any "clustering" at all?

- This seems hard; let's approach the problem from a different point of view

# A Second Perspective—Stick-Breaking

(e.g., Connor & Mosimann; Doksum; Freedman; Kingman; Pitman; Sethuraman)

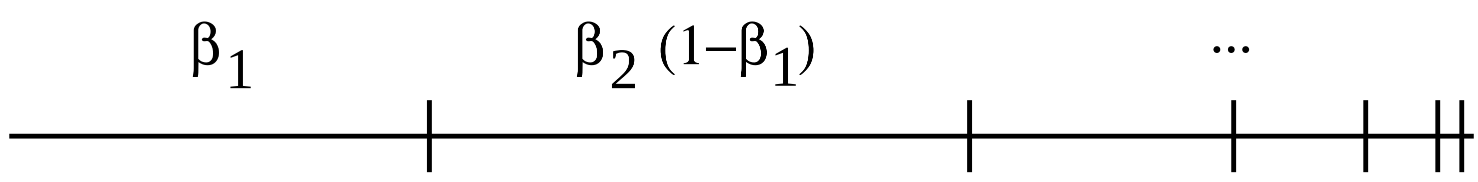
- Define an infinite sequence of Beta random variables:

$$\beta_k \sim \mathrm{Beta}(1, \alpha_0) \qquad\qquad k = 1, 2, \ldots$$

- And then define an infinite sequence of mixing proportions as:

$$\begin{aligned} \pi_1 &= \beta_1 \\ \pi_k &= \beta_k \prod_{l=1}^{k-1} (1 - \beta_l) \qquad\qquad k = 2, 3, \ldots \end{aligned}$$

- This can be viewed as breaking off portions of a stick:



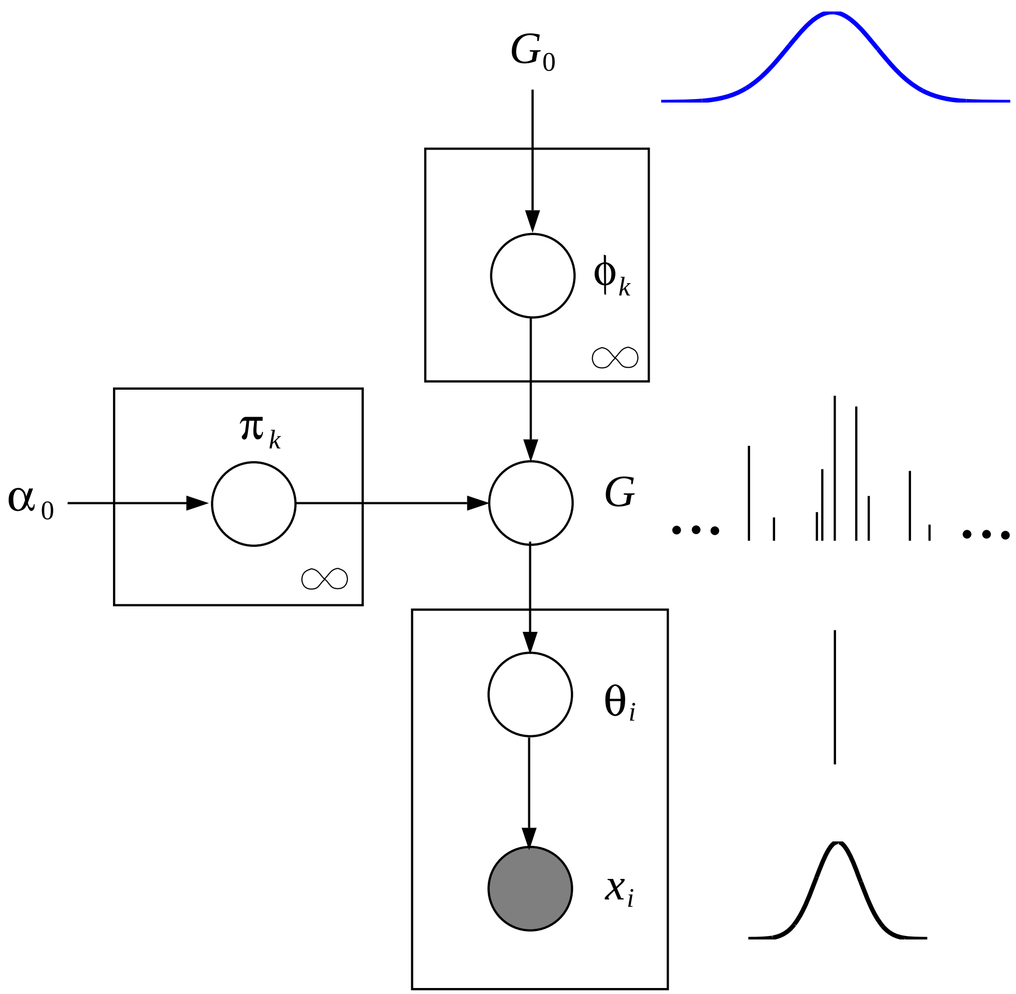
## Stick-Breaking (cont)

- We now have an explicit formula for each $\pi_k$: $\beta_k \prod_{l=1}^{k-1}(1-\beta_l)$
- We can also easily see that $\sum_{k=1}^{\infty} \pi_k = 1$ (wp1):

$$
\begin{aligned}
1 - \sum_{k=1}^{K} \pi_k &= 1 - \beta_1 - \beta_2(1-\beta_1) - \beta_3(1-\beta_1)(1-\beta_2) - \cdots \\
&= (1-\beta_1)(1-\beta_2-\beta_3(1-\beta_2) - \cdots) \\
&= \prod_{k=1}^{K}(1-\beta_k) \\
&\to 0 \qquad (\text{wp1 as } K \to \infty)
\end{aligned}
$$

- So now $G = \sum_{k=1}^{\infty} \pi_k \delta_{\phi_k}$ has a clean definition as a random measure

# Graphical Model Representation



# The Posterior Dirichlet Process

- Suppose that we sample $G$ from a Dirichlet process and then sample $\theta_1$ from $G$. What is the posterior process?

- For a fixed partition, we get a standard Dirichlet update (for the cell that contains $\theta_1$ the exponent increases by one; stays the same for all other cells)
  - this is true for even the tiniest cell
  - suggests that the posterior is a Dirichlet process in which the base measure has an atom at $\theta_1$

- Indeed, we have (for a proof, see, e.g., Schervish, 1995):

$$G \mid \theta_1 \sim \mathrm{DP}(\alpha_0 G_0 + \delta_{\theta_1})$$

- Iterating the posterior update yields:

$$G \mid \theta_1, \ldots, \theta_n \sim \mathrm{DP}(\alpha_0 G_0 + \sum_{i=1}^{n} \delta_{\theta_i})$$

# Relationship to Stick-Breaking

- Recalling the formula for the expectation of a Dirichlet random variable, for any set $A \subseteq \Omega$, we have:

$$\mathbb{E}[G(A) \mid \theta_1, \ldots, \theta_n] = \frac{\alpha_0 G_0(A) + \sum_{i=1}^n \delta_{\theta_i}(A)}{\alpha_0 + n} \rightarrow \sum_{k=1}^{\infty} \pi_k \delta_{\phi_k}(A)$$

  where $\phi_k$ are the unique values of the $\theta_i$, where $\pi_k = \lim_{n\rightarrow\infty} n_k/n$, and where $n_k$ is the number of repeats of $\phi_k$ in the sequence $(\theta_1, \ldots, \theta_n)$

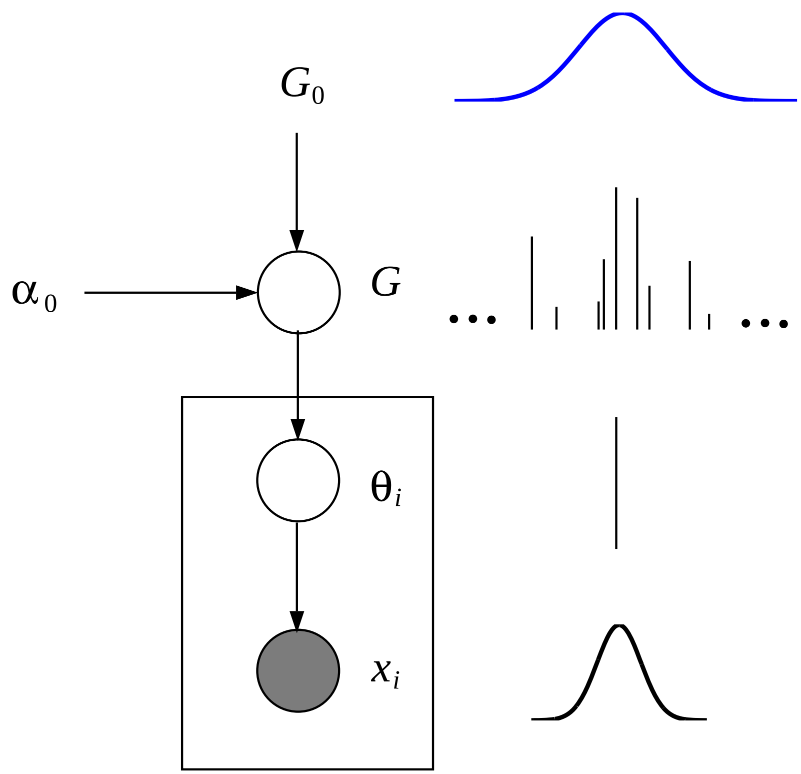
  – assuming that the posterior concentrates, this suggests that the random measures $G \sim \mathrm{DP}(\alpha_0 G_0)$ are discrete (wp1)

- Is there an infinite sum of the form $G = \sum_{k=1}^{\infty} \pi_k \delta_{\phi_k}$ that obeys the definition of the Dirichlet process?

  – yes, the stick-breaking random measure!

  – this important result is not hard to prove; it follows from elementary facts about the Dirichlet distribution (Sethuraman, 1994)

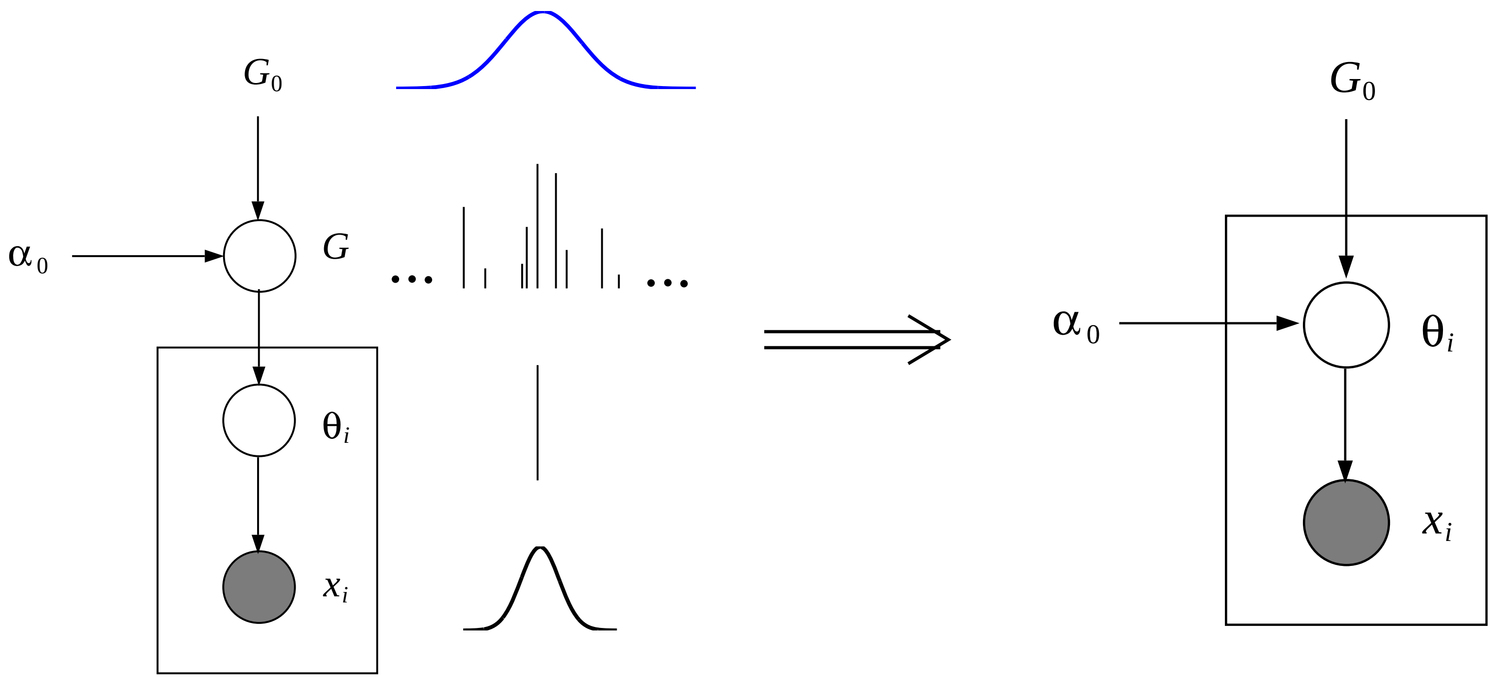
# Dirichlet Process Mixture Models




$$
\begin{aligned}
G &\sim \mathrm{DP}(\alpha_0 G_0) \\
\theta_i \,|\, G &\sim G \qquad\qquad\qquad i \in 1, \ldots, n \\
x_i \,|\, \theta_i &\sim F(x_i \,|\, \theta_i) \qquad i \in 1, \ldots, n
\end{aligned}
$$

# Marginal Probabilities

- To obtain the marginal probability of the parameters $\theta_1, \theta_2, \ldots$, we need to integrate out $G$



# Marginal Probabilities (cont)

- Recall the formula

$$\mathbb{E}[G(A) \mid \theta_1, \ldots, \theta_n] = \frac{\alpha_0 G_0(A) + \sum_{k=1}^{K} n_k \delta_{\phi_k}(A)}{\alpha_0 + n}$$

- Let $A$ be a singleton set equal to one of the $\phi_k$. The formula says that the marginal probability of observing $\phi_k$ again is proportional to $n_k$.

- And the marginal probability of observing a new $\phi$ vector is proportional to $\alpha_0$.

- This is just the Pólya urn scheme!

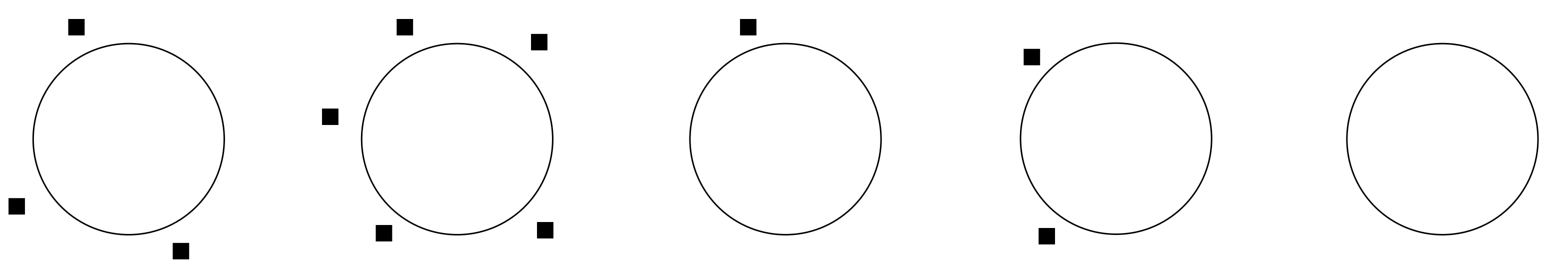
- I.e., integrating over the random measure $G$, where $G \sim \mathrm{DP}(\alpha_0 G_0)$, yields the Pólya urn

# Chinese Restaurant Process (CRP)

- A random process in which $n$ customers sit down in a Chinese restaurant with an infinite number of tables
  - first customer sits at the first table
  - $m$th subsequent customer sits at a table drawn from the following distribution:

$$\begin{aligned} P(\text{previously occupied table } i \,|\, \mathcal{F}_{m-1}) &\propto n_i \\ P(\text{the next unoccupied table} \,|\, \mathcal{F}_{m-1}) &\propto \alpha_0 \end{aligned} \quad (1)$$

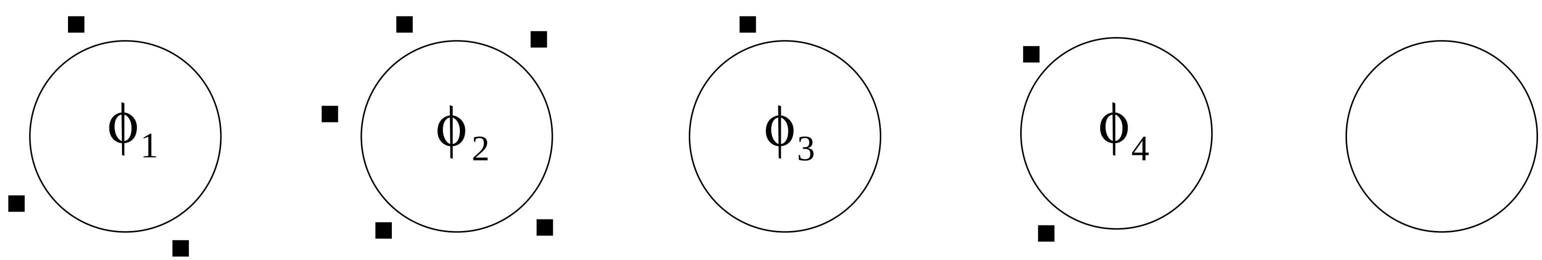
  where $n_i$ is the number of customers currently at table $i$ and where $\mathcal{F}_{m-1}$ denotes the state of the restaurant after $m-1$ customers have been seated



# The CRP and Clustering

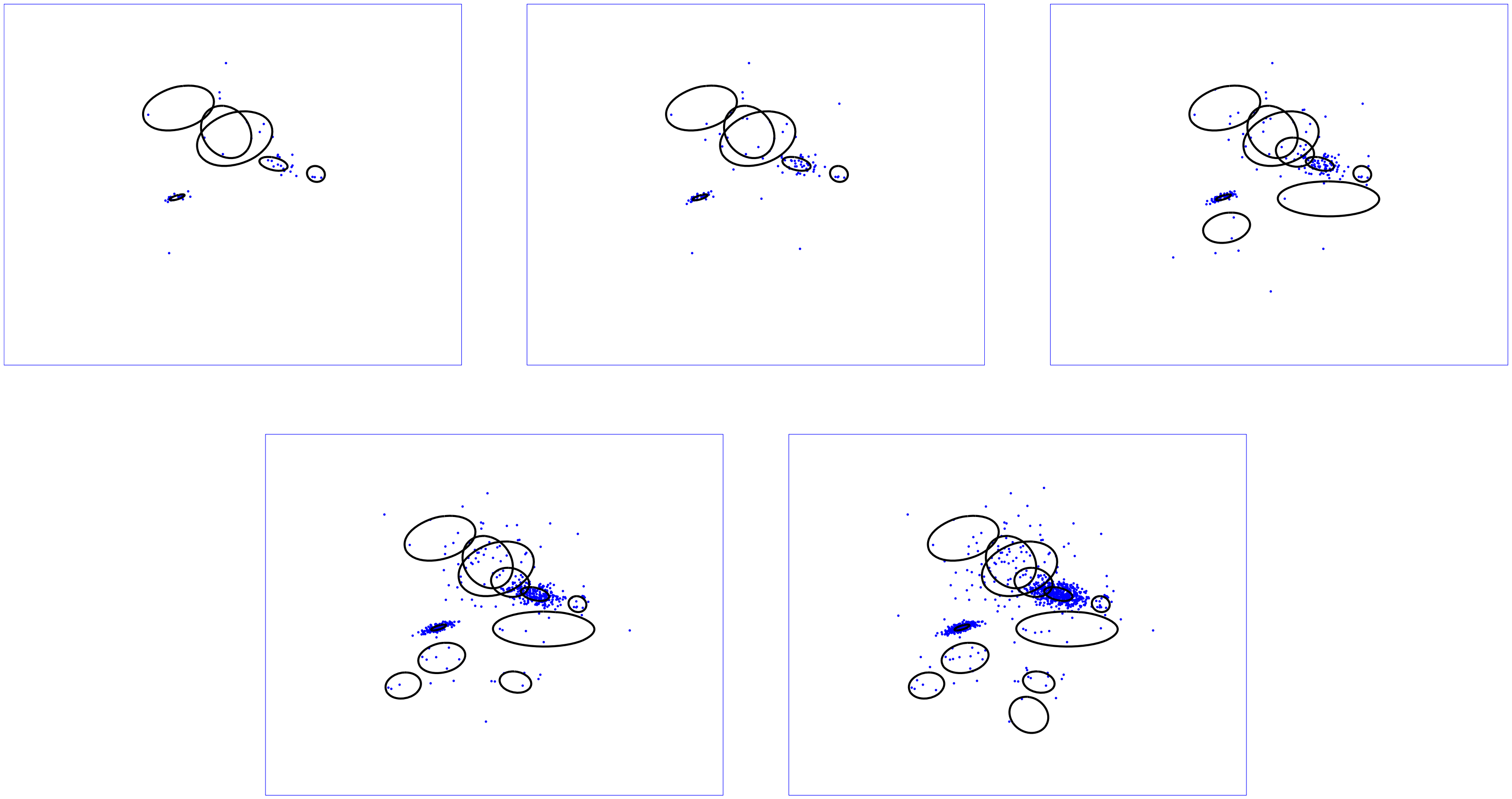
- Data points are customers; tables are clusters
  - the CRP defines a prior distribution on the partitioning of the data and on the number of tables

- This prior can be completed with:
  - a likelihood—e.g., associate a parameterized probability distribution with each table
  - a prior for the parameters—the first customer to sit at table $k$ chooses the parameter vector for that table ($\phi_k$) from the prior




- So we now have a distribution—or can obtain one—for any quantity that we might care about in the clustering setting

# CRP Prior, Gaussian Likelihood, Conjugate Prior



$$\phi_k = (\mu_k, \Sigma_k) \sim N(a,b) \otimes IW(\alpha, \beta)$$

$$x_i \sim N(\phi_k) \qquad \text{for a data point } i \text{ sitting at table } k$$