CSci 8980: Advanced Topics in Graphical Models

MCMC, Gibbs Sampling

Instructor: Arindam Banerjee

September 27, 2007

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- Optimization, Model Selection, etc.

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- For finite σ_f^2, central limit theorem implies

$$\sqrt{n}(I_n(f) - I(f)) \underset{n \to \infty}{\Longrightarrow} \mathcal{N}(0, \sigma_f^2)$$

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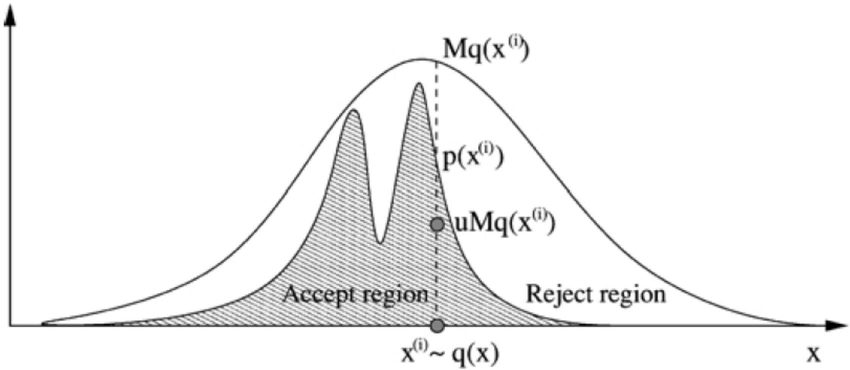
Rejection Sampling (Contd.)



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- The estimator is unbiased, and converges to $I(f)$ a.s.

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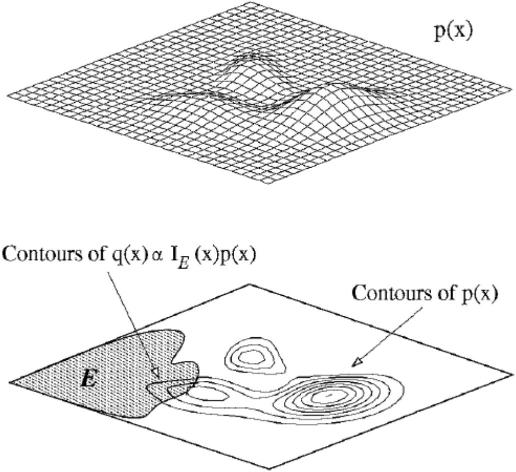

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- Design of samplers for fast convergence

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- $p(x)$ is the corresponding eigenfunction

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  - Then

$$x_{i+1} = \begin{cases} x^* & \text{if} \quad u < A(x_i, x^*) = \min\left\{1, \frac{p(x^*)q(x_i|x^*)}{p(x_i)q(x^*|x_i)}\right\} \\ x_i & \text{otherwise} \end{cases}$$

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- The transition kernel is

$$K_{MH}(x_{i+1}|x_i) = q(x_{i+1}|x_i)A(x_i, x_{i+1}) + \delta_{x_i}(x_{i+1})r(x_i)$$

where $r(x_i)$ is the term associated with rejection

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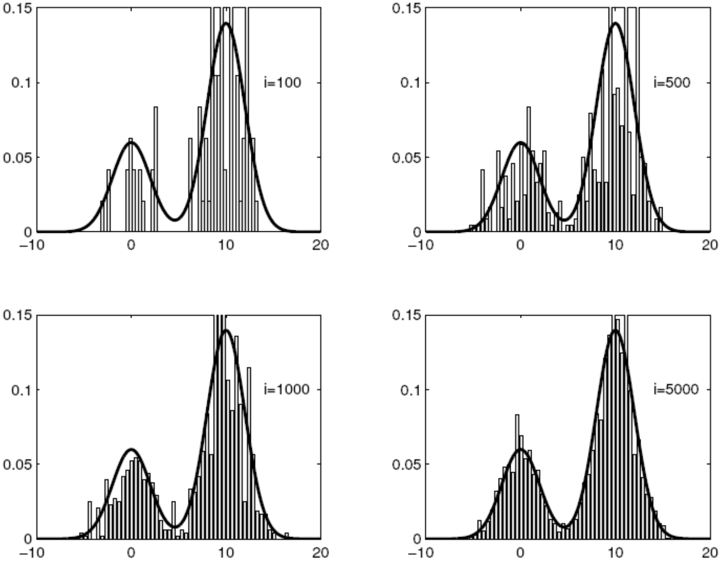

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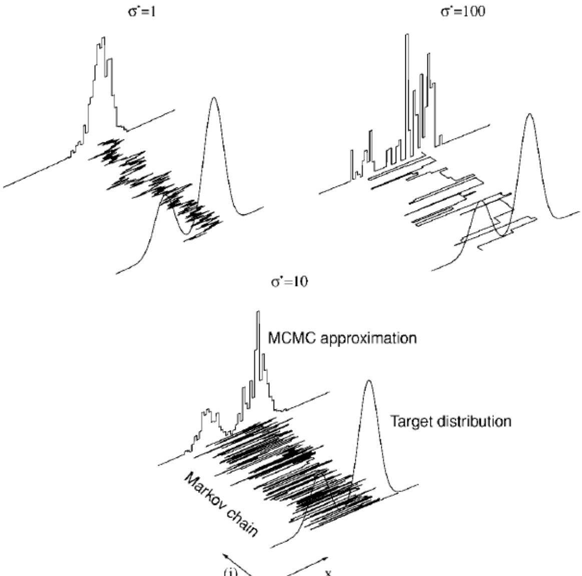

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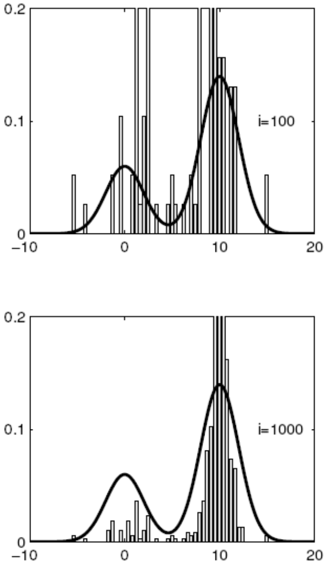

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  - Local proposals get the neighborhood of peaks (random walk)

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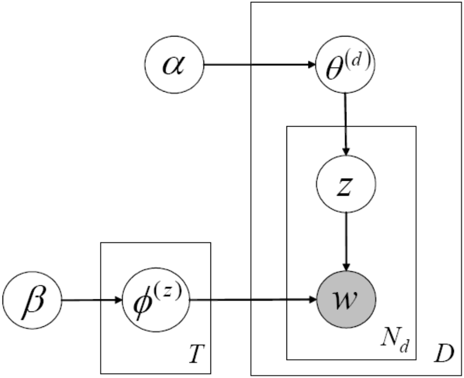
Bayesian LDA



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- When $L = 1$, one obtains the Langevin algorithm

$$x^* = x_0 + \rho u_0 = x^{(i-1)} + \rho(u^* + \rho \Delta(x^{(i-1)})/2)$$

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  - Large ρ gives low acceptance, small ρ needs many steps
  - Large L gives candidates far from x_0, but expensive

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The Slice Sampler (Contd.)