The LASSO

CSci 8980: ML at Large Scale and High Dimensions

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  - Ridge regression: Shrinks coefficients, but not to 0

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  - Some coefficients will become 0

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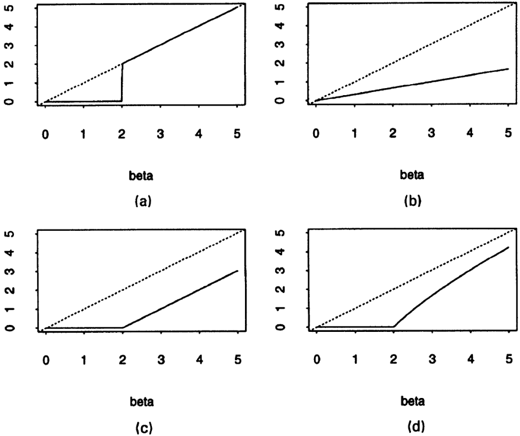
- Garotte estimates

$$\hat{\beta}_j^{\text{garotte}} = \left(1 - \frac{\gamma}{(\hat{\beta}_j^0)^2}\right)_+ \hat{\beta}_j^0$$

Orthonormal Design Case



Shrinkage due to (a) subset selection, (b) ridge regression, (c) the lasso, and (b) the garotte

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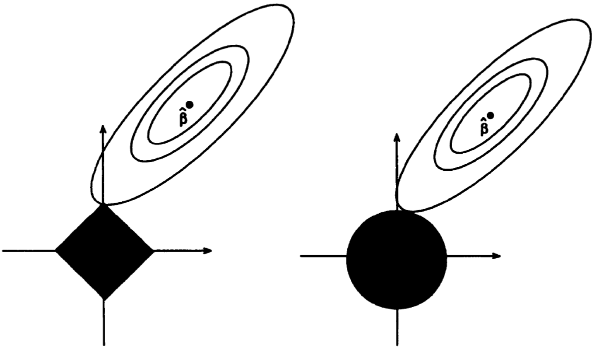
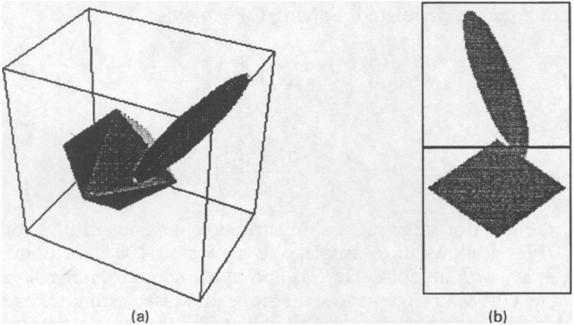
- Ridge vs Lasso: Can the sign change from OLS estimate?

Geometry of LASSO: $p = 2$



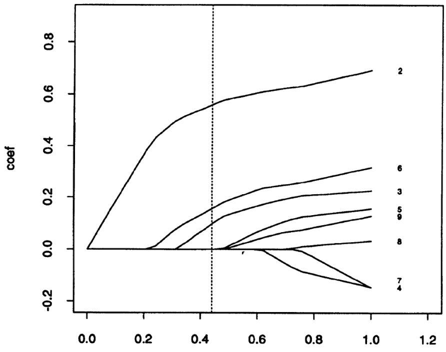
Estimation in (a) the lasso, and (b) ridge regression

Geometry of LASSO: $p > 2$



Sign change in LASSO vs OLS is possible for $p > 2$

Example: Regularization Path



Shrinkage of parameters over $s = \frac{t}{\sum_j \hat{\beta}_j^0}$

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- Resampling based estimates, e.g., stability selection

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- We will consider efficient algorithms for such general problems

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- T_{DP} needs knowledge of β, not practical

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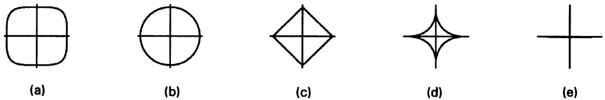
  - Generalized to other sparsity inducing norms

Norm level sets



L_q norm level sets: (a) $q = 4$, (b) $q = 2$, (c) $q = 1$, (d) $q = 0.5$, (e) $q = 0.1$