The Convex Geometry of Linear Inverse Problems.

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CSCI 8990
ML at Large Scale and High Dimensions

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Reference

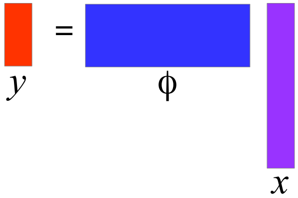
V. Chandrasekaran, B. Recht, P. A. Parrilo, and A. S. Willsky
"The Convex Geometry of Linear Inverse Problems."
Foundations of Computational Mathematics, 12, 805-849, 2012.

1. Introduction
2. Unified Convex optimization Framework
3. Recovery Condition
   - Number of required measurements for unique true recovery
4. Computational Issues
5. Noisy Scenario

Inverse Problem

- Find the solution of

$$\mathbf{y} = \phi \mathbf{x} \quad \phi \in \mathbb{R}^{m \times n}$$

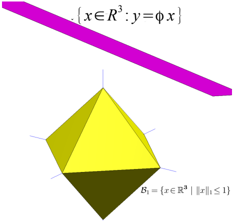


- Given $\mathbf{y} \Rightarrow$ recover $\mathbf{x}$.
- Limited Linear Measurements: ill posed problem
- Infinite solution, which one to choose?
- Examples
  - Sparse vectors: signal processing, statistics
  - Low-rank matrices: control, statistics, collaborative filtering
  - Sums of a few permutation matrices: ranked elections, multiobject tracking
  - Low-rank tensors: computer vision, neuroscience
  - Orthogonal matrices: machine learning

Review: Sparsity and Low-rank

- Minimizing ℓ_1 norm, yields sparse solution

$$\min_{\mathbf{x}} \| x \|_1 \quad s.t. \quad \mathbf{y} = \phi \mathbf{x}$$



- Minimizing nuclear norm, yields low rank solution
- Both are convex problem, can be solved efficiently
- Can be generalized?

Convex Optimization Framework

- Simple Models from atomic set A

Rank

$$x = \sum_{i=1}^{r} c_i a_i \qquad a_i \in A, \quad c_i \geq 0$$

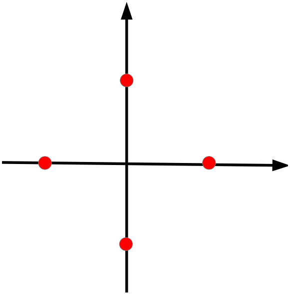
Model — Weights — Atoms

- Atomic norm induced by convex hull of A

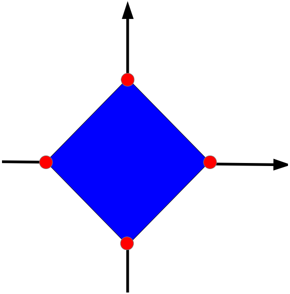
$$\| x \|_A = \inf\{t > 0 : x \in t\,\mathrm{conv}(A)\}$$

$$\| x \|_A = \inf\left\{\sum_i c_i : x = \sum_{i=1}^{r} c_i \mathbf{a}_i, c_i \geq 0, \mathbf{a}_i \in A\right\}$$

Geometric view - Sparsity



$\Rightarrow$

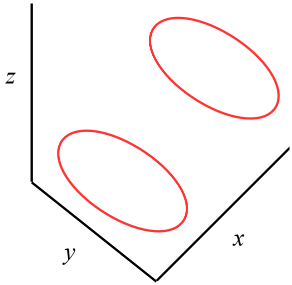


1-sparse vectors of
Euclidean norm 1

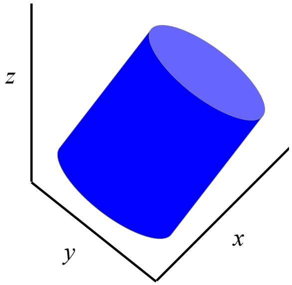
Convex Hull: ℓ_1 norm

$$\| \mathbf{x} \|_1 = \sum_{i=1}^{n} |x_i|$$

Geometric view - Low rank



$$\begin{pmatrix} x & y \\ y & z \end{pmatrix} \Rightarrow$$



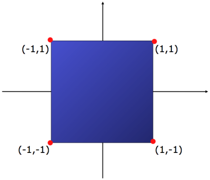
2×2 rank 1 symmetric matrices (normalized)

Convex Hull: nuclear norm

$$\| \mathbf{X} \|_* = \sum_i \sigma_i(X)$$

Other Convex Hulls

Sign Vectors

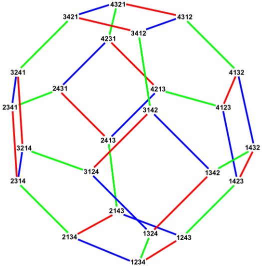


- Hypercube polytope
- Integer Programming

Cut Matrices

- Cut polytope
- Atoms: rank-1 sign matrices

Permutation Matrices



- Birkhoff polytope
- Permutahedra
- Ranking context
- Object tracking context

Convex Optimization Framework

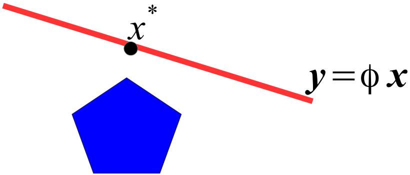
- Consider true $\mathbf{x}^*$ concise w.r.t to atomic set A
- Given linear measurement $\mathbf{y} = \phi \mathbf{x}^*$, solve

$$\hat{\mathbf{x}} = \arg\min_{\mathbf{x}} \| x \|_A \quad \text{s.t.} \quad \mathbf{y} = \phi \mathbf{x}$$

- Recovery condition?

Recovery Condition: Geometric View

- When does $\hat{\mathbf{x}} = \mathbf{x}^*$?



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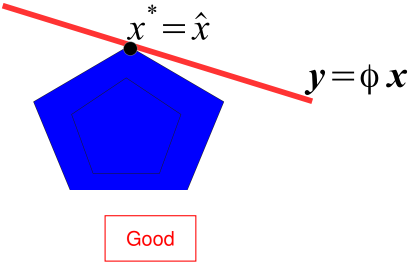

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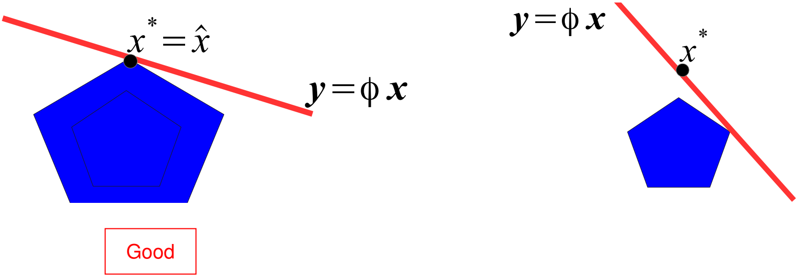

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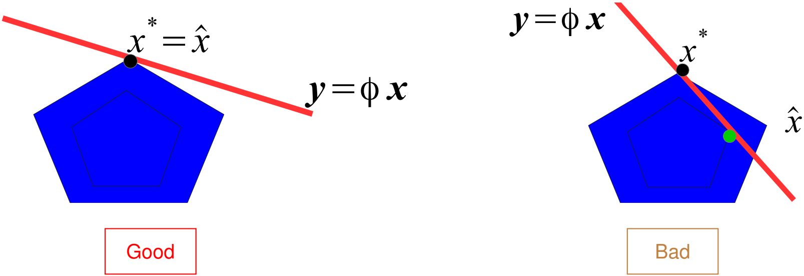

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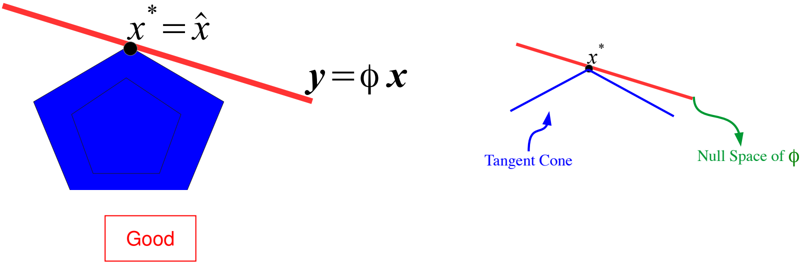


Recovery Condition

- Tangent Cone at x:

$$T_A(\mathbf{x}) = \{\mathbf{z} - \mathbf{x} : \| z \|_A \leq \| x \|_A\}$$

- Set of descent directions of $\| \, . \, \|_A$ at point $\mathbf{x}$.

Proposition 2.1

$$\hat{\mathbf{x}} = \mathbf{x}^* \iff null(\phi) \cap T_A(\mathbf{x}^*) = \{0\}$$

- Why Atomic Norm?

Recovery from Generic Measurements

- Number of measurements n for exact recovery?
- Gaussian Width:

$$w(S) := \mathbb{E}_{\mathbf{g}} \left[\sup_{\mathbf{z} \in S \cap \mathcal{B}(0,1)} \mathbf{g}^T \mathbf{z} \right]$$

- $\mathbf{g} \sim \mathcal{N}(0, I)$
- $\mathcal{B}(0, 1)$: Unit Euclidean ball.

Corollary 3.3

- $\mathbf{y} = \phi \mathbf{x}^*$
- $\phi : \mathbb{R}^p \to \mathbb{R}^n$ i.i.d. zero-mean Gaussian entries
- $\hat{\mathbf{x}} = \mathbf{x}^*$ W.H.P. if

$$n \geq w(T_A(\mathbf{x}^*))^2 + 1$$

- Gordon 1988

proof of Corollary 3.3

- $\| \phi \mathbf{z} \|$ Minimum gain of the ϕ restricted to $T_A(\mathbf{x}^*)$
- Bounding $\| \phi \mathbf{z} \|$ away from zero $\quad \mathbf{z} \in T_A(\mathbf{x}^*)$

Theorem 3.2

- Restricted minimum singular values
- λ_n: expected length of a n-dimensional Gaussian vector
- $\frac{n}{\sqrt{n+1}} \leq \lambda_n \leq \sqrt{n}$
- Ω: Closed subset of unit sphere $\mathbb{S}^{p-1}$
- $\phi : \mathbb{R}^p \leftarrow \mathbb{R}^n$: random map with i.i.d Gaussian entries

$$\mathbb{E}\left[\min_{\mathbf{z}\in\Omega} \| \phi \mathbf{z} \|_2\right] \geq \lambda_n - w(\Omega)$$

- Gordon1988

proof of Corollary 3.3

- $\mathbf{g} \sim \mathcal{N}(0, I)$
- f be Lipschitz constant L

$$P(f(\mathbf{g}) \geq \mathbb{E}[f] - t) \geq 1 - \exp(-\frac{t^2}{2L^2})$$

- $\min_{\mathbf{z}\in\Omega} \parallel \phi\mathbf{z} \parallel_2$ is 1-Lipschitz
- $\mathbb{E}\left[\min_{\mathbf{z}\in\Omega} \parallel \phi\mathbf{z} \parallel_2\right] \geq \lambda_n - w(\Omega)$

$$P(\min_{\mathbf{z}\in\Omega} \parallel \phi\mathbf{z} \parallel_2 \geq \epsilon) \geq 1 - \exp(-\frac{1}{2}(\lambda_n - w(\Omega) - \sqrt{n}\epsilon)^2)$$

$$\geq 0$$

- Set $\epsilon = 0 \quad \Rightarrow \quad \lambda_n \geq w(\Omega)$
- $w(\Omega) \leq \lambda_n \leq \sqrt{n}$

Gaussian Width via Dual Cone

- Gaussian width of a cone via the distance to the dual cone
- Polar cone of $\mathcal{C}$:

$$\mathcal{C}^* = \{\mathbf{x} \in \mathbb{R}^p : \langle \mathbf{x}, \mathbf{z} \rangle \leq 0 \ \ \forall \mathbf{z} \in \mathcal{C}\}$$

Proposition 3.6

- $\mathbf{g} \sim \mathcal{N}(0, I)$
- dist: Euclidean distance of a point to a set

$$w(\mathcal{C}) \leq \mathbb{E}_{\mathbf{g}} \left[\mathrm{dist}(\mathbf{g}, \mathcal{C}^*)\right]$$

$$w(\mathcal{C})^2 \leq \mathbb{E}_{\mathbf{g}} \left[\mathrm{dist}(\mathbf{g}, \mathcal{C}^*)^2\right]$$

Proof

- Gaussian Width: $w(\mathcal{C} \cap \mathbb{S}^{p-1}) \leq \mathbb{E}_{\mathbf{g}} \left[\sup_{\mathbf{z} \in \mathcal{C} \cap \mathcal{B}(0,1)} \mathbf{g}^T \mathbf{z} \right]$
- Inside the expected value is the optimal solution to

$$\max_{z} \mathbf{g}^T \mathbf{z} \quad \text{s.t.} \quad \mathbf{z} \in \mathcal{C}, \quad \| z \|^2 \leq 1$$

- Introducing the Lagrangian:

$$\mathcal{L}(\mathbf{z}, \mathbf{u}, \gamma) = \mathbf{g}^T \mathbf{z} + \gamma(1 - \mathbf{z}^T \mathbf{z}) - \mathbf{u}^T \mathbf{z}$$

- minimize w.r.t $\mathbf{z}$ and γ

$$\mathbf{z} = \frac{1}{2\gamma}(\mathbf{g} - \mathbf{u}) \qquad \gamma = \frac{1}{2} \| \mathbf{g} - \mathbf{u} \|$$

- Dual Problem:

$$\min \| \mathbf{g} - \mathbf{u} \| \quad \text{s.t} \quad \mathbf{u} \in \mathcal{C}^*$$

Properties of $w(\mathcal{C})$

- **Lemma 3.7**: $\mathcal{C} \subset \mathbb{R}^p$, $\quad w(\mathcal{C})^2 + w(\mathcal{C}^*)^2 \leq p$
- proof:
- $\mathbf{g} = \sqcap_{\mathcal{C}}(\mathbf{g}) + \sqcap_{\mathcal{C}^*}(\mathbf{g})$ where $\langle \sqcap_{\mathcal{C}}(\mathbf{g}), \sqcap_{\mathcal{C}^*}(\mathbf{g}) \rangle = 0$
- $\text{dist}(\mathbf{g}, \mathcal{C}) = \| \sqcap_{\mathcal{C}^*}(\mathbf{g}) \|$

$$
\begin{aligned}
w(\mathcal{C})^2 &\leq \mathbb{E}_{\mathbf{g}}[\text{dist}(\mathbf{g}, \mathcal{C}^*)^2] \\
&= \mathbb{E}_{\mathbf{g}}[\| \mathbf{g} \|^2 - \| \sqcap_{\mathcal{C}^*}(\mathbf{g}) \|^2] = p - \mathbb{E}_{\mathbf{g}}[\text{dist}(\mathbf{g}, \mathcal{C})^2] \\
&\leq p - w(\mathcal{C}^*)^2
\end{aligned}
$$

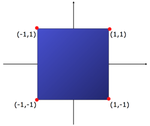
- **Corollary 3.8**: Self dual cone $\mathcal{C} = -\mathcal{C}^*$

$$
w(\mathcal{C})^2 \leq \frac{p}{2}
$$

Special Cases

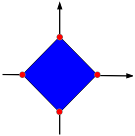
- Hypercube:

$$w(T_A(\mathbf{x}^*))^2 \leq \frac{p}{2}$$



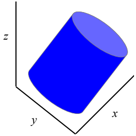
- s-sparse vector $\mathbf{x}^* \in \mathbb{R}^p$:

$$w(T_A(\mathbf{x}^*))^2 \leq 2s\log(\frac{p}{s}) + \frac{5}{4}s$$



- Low-rank matrices $\in \mathbb{R}^{m_1 \times m_2}$, rank r

$$w(T_A(\mathbf{x}^*))^2 \leq 3r(m_1 + m_2 - r)$$



General Cones

Theorem 3.9

- $\mathcal{C} \subseteq \mathbb{R}^p$: close, convex, solid cone
- $\mathcal{C}^*$: has volume of $\theta \in [0, 1]$

$$w(\mathcal{C}) \leq 3\sqrt{\log \frac{4}{\theta}}$$

- **Corollary 3.14**
  For a symmetric polytope with m vertices

$$n \geq O(\log m)$$

Approximations

- Atomic set A are Algebraic variety
- Well-approximated in a constructive manner by
  - linear matrix inequality constraints
- Semidefinite representations are intractable?
  - Hierarchy of tractable semidefinite relaxations

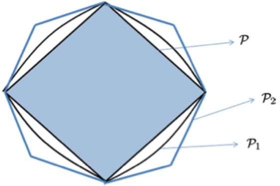
Complexity vs Number of Measurements

- Intractable to compute norm induced by cut Polytope:

$$\mathcal{P} = \text{conv}\{\mathbf{z}^T\mathbf{z} : \mathbf{z} \in \{-1, +1\}^m\}$$

- MAX-CUT problem
- Semidefinite relaxation:
  $\mathcal{P}_1 = \{\mathcal{M} : \mathcal{M} \ \text{Symmetric}, \ \ \mathcal{M} \succcurlyeq 0, \mathcal{M}_{ii} = 1\}$
- Trivial hypercube relaxation:
  $\mathcal{P}_2 = \{\mathcal{M} : \mathcal{M} \ \text{Symmetric}, \ \ \mathcal{M}_{ii} = 1, \ \ |\mathcal{M}_{ij}| < 1 \ \ \forall i \neq j\}$

- Using $\mathcal{P} : n = O(m)$
- Using $\mathcal{P}_1 : n = O(m)$
- Using $\mathcal{P}_2 : n = O(\frac{m^2 - m}{4})$



Robust Recovery

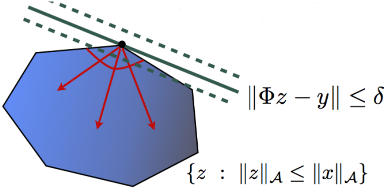
- Noisy Scenario: $\mathbf{y} = \phi \mathbf{x}^* + \omega \qquad \| \omega \| \leq \delta$

$$\hat{\mathbf{x}} = \arg\min_{\mathbf{x}} \| \mathbf{x} \|_A$$
$$\text{s.t.} \quad \| \mathbf{y} - \phi \mathbf{x} \| \leq \delta$$

- Robust recovery: $\| \mathbf{x}^* - \hat{\mathbf{x}} \| \leq \frac{2\delta}{\epsilon}$ W.H.P. provided by

$$n \geq \frac{c_0 w(T_A(\mathbf{x}^*))^2}{(1-\epsilon)^2}$$

- $\| \phi \mathbf{z} \| \geq \epsilon \| \mathbf{z} \|$



Conclusion

- Providing a unified convex optimization framework for Inverse problem
- Recovery condition
    - Noiseless scenario
    - Noisy scenario
- Number of measurements for true unique recovery
- Tradeoff: complexity and number of measurements

Questions?