

Parallel Direction Method of Multipliers

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Abstract

We consider the problem of minimizing block-separable convex functions subject to linear constraints. While the Alternating Direction Method of Multipliers (ADMM) for two-block linear constraints has been intensively studied both theoretically and empirically, in spite of some preliminary work, effective generalizations of ADMM to multiple blocks is still unclear. In this paper, we propose a randomized block coordinate method named Parallel Direction Method of Multipliers (PDMM) to solve the optimization problems with multi-block linear constraints. PDMM randomly updates some primal blocks in parallel, behaving like parallel randomized block coordinate descent. We establish the global convergence and the iteration complexity for PDMM with constant step size. We also show that PDMM can do randomized block coordinate descent on overlapping blocks. Experimental results show that PDMM performs better than state-of-the-arts methods in two applications, robust principal component analysis and overlapping group lasso.

1 Introduction

In this paper, we consider the minimization of block-separable convex functions subject to linear constraints, with a canonical form:

$$\min_{\{\mathbf{x}_j \in \mathcal{X}_j\}} f(\mathbf{x}) = \sum_{j=1}^J f_j(\mathbf{x}_j), \text{ s.t. } \mathbf{A}\mathbf{x} = \sum_{j=1}^J \mathbf{A}_j^c \mathbf{x}_j = \mathbf{a}, \quad (1)$$

where the objective function $f(\mathbf{x})$ is a sum of J block separable (nonsmooth) convex functions, $\mathbf{A}_j^c \in \mathbb{R}^{m \times n_j}$ is the j -th column block of $\mathbf{A} \in \mathbb{R}^{m \times n}$ where $n = \sum_j n_j$, $\mathbf{x}_j \in \mathbb{R}^{n_j \times 1}$ is the j -th block coordinate of \mathbf{x} , \mathcal{X}_j is a local convex constraint of \mathbf{x}_j and $\mathbf{a} \in \mathbb{R}^{m \times 1}$. The canonical form can be extended to handle linear inequalities by introducing slack variables, i.e., writing $\mathbf{A}\mathbf{x} \leq \mathbf{a}$ as $\mathbf{A}\mathbf{x} + \mathbf{z} = \mathbf{a}$, $\mathbf{z} \geq \mathbf{0}$.

A variety of machine learning problems can be cast into the linearly-constrained optimization problem (1). For example, in robust Principal Component Analysis (RPCA) [5], one attempts to recover a low rank matrix \mathbf{L} and a sparse matrix \mathbf{S} from an observation matrix \mathbf{M} , i.e., the linear constraint is $\mathbf{M} = \mathbf{L} + \mathbf{S}$. Further, in the stable version of RPCA [43], a noisy matrix \mathbf{Z} is taken into consideration, and the linear constraint has three blocks, i.e., $\mathbf{M} = \mathbf{L} + \mathbf{S} + \mathbf{Z}$. The linear constraint with three blocks also appears in the

latent variable Gaussian graphical model selection problem [6, 23]. Problem (1) can also include composite minimization problems which solve a sum of a loss function and a set of nonsmooth regularization functions. Due to the increasing interest in structural sparsity [2], composite regularizers have become widely used, e.g., overlapping group lasso [42]. As the blocks are overlapping in this class of problems, it is difficult to apply block coordinate descent methods for large scale problem [24, 27] which assume block-separable. By simply splitting blocks through introducing equality constraints, the composite minimization problem can also be formulated as (1) [3].

A classical approach to solving (1) is to relax the linear constraints using the (augmented) Lagrangian [28, 29], i.e.,

$$L_\rho(\mathbf{x}, \mathbf{y}) = f(\mathbf{x}) + \langle \mathbf{y}, \mathbf{Ax} - \mathbf{a} \rangle + \frac{\rho}{2} \|\mathbf{Ax} - \mathbf{a}\|_2^2. \quad (2)$$

where $\rho \geq 0$ is called the penalty parameter. We call \mathbf{x} the primal variable and \mathbf{y} the dual variable. (2) usually leads to primal-dual algorithms which update the primal and dual variables alternatively. The dual update is simply dual gradient ascent where the dual gradient is the residual of equality constraint, i.e., $\mathbf{Ax} - \mathbf{a}$. The primal update is to solve a minimization problem of (2) given \mathbf{y} . The primal update determines the efficiency of this class of primal-dual algorithms and will be the focus of this paper.

If $\rho = 0$, (2) decomposes into J independent subproblems provided f is separable. In this scenario, the primal-dual algorithm is called the dual ascent method [4, 31], where the primal update is solved in a parallel block coordinate fashion. While the dual ascent method can achieve massive parallelism, a careful choice of stepsize and some strict conditions are required for convergence, particularly when f is nonsmooth. To achieve better numerical efficiency and convergence behavior compared to the dual ascent method, it is favorable to set $\rho > 0$ in the augmented Lagrangian (2). However, (2) is no longer separable since the augmentation term makes \mathbf{x} coupled. A well-known primal-dual algorithm to solve (2) is the method of multipliers, which solves the primal update in one block. For large scale optimization problems, it is often difficult to solve the entire augmented Lagrangian efficiently. Considerable efforts have thus been devoted to solving the primal update of the method of multipliers efficiently. In [34], randomized block coordinate descent (RBCD) [24, 27] is used to solve (2) exactly, but leading to a double-loop algorithm along with the dual step. More recent results show (2) can be solved inexactly by just sweeping the coordinates once using the alternating direction method of multipliers (ADMM) [14, 3].

When $J = 2$, the constraint is of the form $\mathbf{A}_1^c \mathbf{x}_1 + \mathbf{A}_2^c \mathbf{x}_2 = \mathbf{a}$. In this case, a well-known variant of the method of multipliers is the Alternating Direction Method of Multipliers (ADMM) [3], which solves the augmented Lagrangian separately and alternatively. ADMM was first introduced in [14] and became popular in recent years due to its ease of applicability and superior empirical performance in a wide variety of applications, ranging from image processing [11, 1, 15] to applied statistics and machine learning [30, 40, 39, 22, 35, 12, 21, 37]. For further understanding of ADMM with two blocks, we refer the readers to the comprehensive review by [3]. The proof of global convergence of ADMM with two blocks can be found in [13, 3]. Recently, it has been shown that ADMM converges at a rate of $O(1/T)$ [35, 18], where T is the number of iterations. For strongly convex functions, the dual objective of an accelerated version of ADMM can converge at a rate of $O(1/T^2)$ [16]. For strongly convex functions, ADMM can achieve a linear convergence rate [10].

Encouraged by the success of ADMM with two blocks, ADMM has also been extended to solve the problem with multiple blocks [20, 19, 9, 26, 17, 7]. The variants of ADMM can be mainly divided into two categories. One is Gauss-Seidel ADMM (GSADMM) [20, 19], which solves (2) in a cyclic block coordinate manner. [20] established a linear convergence rate for MADMM under some fairly strict conditions: (1) \mathbf{A}_j has full column rank; (2) f_j has Lipschitz-continuous gradients; (3) certain local error bounds hold; (4) the

step size needs to be sufficiently small. In [17], a back substitution step was added so that the convergence of ADMM for multiple blocks can be proved. In some cases, it has been shown that ADMM might not converge for multiple blocks [7]. In [19], a block successive upper bound minimization method of multipliers (BSUMM) is proposed to solve the problem (1). The convergence of BSUMM is established under conditions: (i) certain local error bounds hold; (ii) the step size is either sufficiently small or decreasing. However, in general, Gauss-Seidel ADMM with multiple blocks is not well understood and its iteration complexity is largely open. The other is Jacobi ADMM [38, 9, 26], which solves (2) in a parallel block coordinate fashion. In [38, 26], (1) is solved by using two-block ADMM with splitting variables (sADMM). [9] considers a proximal Jacobian ADMM (PJADMM) by adding proximal terms. In addition to the two types of extensions, a randomized block coordinate variant of ADMM named RBSUMM was proposed in [19]. However, RBSUMM can only randomly update one block. Moreover, the convergence of RBSUMM is established under the same conditions as BSUMM and its iteration complexity is unknown. In [32], ADMM with stochastic dual coordinate ascent is proposed to solve online or stochastic ADMM [35, 25, 33] problem in the dual, which is not the focus of this paper.

In this paper, we propose a randomized block coordinate method named parallel direction method of multipliers (PDMM) which randomly picks up any number of blocks to update in parallel, behaving like randomized block coordinate descent [24, 27]. Like the dual ascent method, PDMM solves the primal update in a parallel block coordinate fashion even with the augmentation term. Moreover, PDMM inherits the merits of the method of multipliers and can solve a fairly large class of problems, including nonsmooth functions. Technically, PDMM has three aspects which make it distinct from such state-of-the-art methods. First, if block coordinates of the primal \mathbf{x} is solved exactly, PDMM uses a backward step on the dual update so that the dual variable makes conservative progress. Second, the sparsity of \mathbf{A} and the number of blocks K to be updated are taken into consideration to determine the step size of the dual update. Third, PDMM can randomly choose arbitrary number of primal blocks for update in parallel. Moreover, we show that sADMM and PJADMM are the two extreme cases of PDMM. The connection between sADMM and PJADMM through PDMM provides better understanding of dual backward step. PDMM can also be used to solve overlapping groups in a randomized block coordinate fashion. Interestingly, the corresponding problem for RBCD [24, 27] with overlapping blocks is still an open problem. We establish the global convergence and $O(1/T)$ iteration complexity of PDMM with constant step size. We evaluate the performance of PDMM in two applications: robust principal component analysis and overlapping group lasso.

The rest of the paper is organized as follows. PDMM is proposed in Section 2. The convergence results are established in Section 3. We evaluate the performance of PDMM in Section 5 and conclude the paper in Section 6. The proof of the convergence of PDMM is given in the Appendix.

Notations: Assume that $\mathbf{A} \in \mathbb{R}^{m \times n}$ is divided into $I \times J$ blocks. Let $\mathbf{A}_i^r \in \mathbb{R}^{m_i \times n}$ be the i -th row block of \mathbf{A} , $\mathbf{A}_j^c \in \mathbb{R}^{m \times n_j}$ be the j -th column block of \mathbf{A} , and $\mathbf{A}_{ij} \in \mathbb{R}^{m_i \times n_j}$ be the ij -th block of \mathbf{A} . Let $\mathbf{y}_i \in \mathbb{R}^{m_i \times 1}$ be the i -th block coordinate of $\mathbf{y} \in \mathbb{R}^{m \times 1}$. $\mathcal{N}(i)$ is a set of nonzero blocks \mathbf{A}_{ij} in the i -th row block \mathbf{A}_i^r and $d_i = |\mathcal{N}(i)|$ is the number of nonzero blocks. λ_{\max}^{ij} is the largest eigenvalue of $\mathbf{A}_{ij}^T \mathbf{A}_{ij}$. $\text{diag}(\mathbf{x})$ denotes a diagonal matrix of vector \mathbf{x} . \mathbf{I}_n is an identity matrix of size $n \times n$. Let $\tilde{K}_i = \min\{d_i, K\}$ where K is the number of blocks randomly chosen by PDMM and T be the number of iterations.

Table 1: Parameters (τ_i, ν_i) of PDMM. K is the number of blocks randomly chosen from J blocks, and $\tilde{K}_i = \min\{d_i, K\}$ where d_i is the number of nonzero blocks \mathbf{A}_{ij} in the i -th row of \mathbf{A} .

K	ν_i	τ_i
$K = 1$	0	$\frac{1}{2J-1}$
$1 < K < J$	$1 - \frac{1}{\tilde{K}_i}$	$\frac{K}{\tilde{K}_i(2J-K)}$
$K = J$	$1 - \frac{1}{d_i}$	$\frac{1}{d_i}$

2 Parallel Direction Method of Multipliers

Consider a direct Jacobi version of ADMM which updates all blocks in parallel:

$$\mathbf{x}_j^{t+1} = \operatorname{argmin}_{\mathbf{x}_j \in \mathcal{X}_j} L_\rho(\mathbf{x}_j, \mathbf{x}_{k \neq j}^t, \mathbf{y}^t), \quad (3)$$

$$\mathbf{y}^{t+1} = \mathbf{y}^t + \tau \rho(\mathbf{A}\mathbf{x}^{t+1} - \mathbf{a}). \quad (4)$$

where τ is a shrinkage factor for the step size of the dual gradient ascent update. However, empirical results show that it is almost impossible to make the direct Jacobi updates (3)-(4) to converge even when τ is extremely small. [20, 9] also noticed that the direct Jacobi updates may not converge.

To address the problem in (3) and (4), we propose a backward step on the dual update. Moreover, instead of updating all blocks, the blocks \mathbf{x}_j will be updated in a parallel randomized block coordinate fashion. We call the algorithm Parallel Direction Method of Multipliers (PDMM). PDMM first randomly select K blocks denoted by set \mathbb{I}_t at time t , then executes the following iterates:

$$\mathbf{x}_{j_t}^{t+1} = \operatorname{argmin}_{\mathbf{x}_{j_t} \in \mathcal{X}_{j_t}} L_\rho(\mathbf{x}_{j_t}, \mathbf{x}_{k \neq j_t}^t, \hat{\mathbf{y}}^t) + \eta_{j_t} B_{\phi_{j_t}}(\mathbf{x}_{j_t}, \mathbf{x}_{j_t}^t), \quad j_t \in \mathbb{I}_t, \quad (5)$$

$$\mathbf{y}_i^{t+1} = \mathbf{y}_i^t + \tau_i \rho(\mathbf{A}_i \mathbf{x}^{t+1} - \mathbf{a}_i), \quad (6)$$

$$\hat{\mathbf{y}}_i^{t+1} = \mathbf{y}_i^{t+1} - \nu_i \rho(\mathbf{A}_i \mathbf{x}^{t+1} - \mathbf{a}_i), \quad (7)$$

where $\tau_i > 0, 0 \leq \nu_i < 1, \eta_{j_t} \geq 0$, and $B_{\phi_{j_t}}(\mathbf{x}_{j_t}, \mathbf{x}_{j_t}^t)$ is a Bregman divergence. Note $\mathbf{x}^{t+1} = (\mathbf{x}_{j_t}^{t+1}, \mathbf{x}_{k \neq j_t}^t)$ in (6) and (7). Table 1 shows how to choose τ_i and ν_i under different number of random blocks K and block sparsity of \mathbf{A} . K is the number of blocks randomly chosen from J blocks, and $\tilde{K}_i = \min\{d_i, K\}$ where d_i is the number of nonzero blocks \mathbf{A}_{ij} in the i -th row of \mathbf{A} .

In the \mathbf{x}_{j_t} -update (5), a Bregman divergence is added so that exact PDMM and its inexact variants can be analyzed in a unified framework [36]. In particular, if $\eta_{j_t} = 0$, (5) is an exact update. If $\eta_{j_t} > 0$, by choosing a suitable Bregman divergence, (5) can be solved by various inexact updates, often yielding a closed-form for the \mathbf{x}_{j_t} update (see Section 2.1).

Let $\mathbf{r}^t = \mathbf{A}\mathbf{x}^t - \mathbf{a}$, then $\mathbf{r}^{t+1} = \mathbf{r}^t + \sum_{j_t \in \mathbb{I}_t} \mathbf{A}_{j_t}^c (\mathbf{x}_{j_t}^{t+1} - \mathbf{x}_{j_t}^t)$. (5) can be rewritten as

$$\begin{aligned} \mathbf{x}_{j_t}^{t+1} &= \operatorname{argmin}_{\mathbf{x}_{j_t} \in \mathcal{X}_{j_t}} f_{j_t}(\mathbf{x}_{j_t}) + \langle \hat{\mathbf{y}}^t, \mathbf{A}_{j_t}^c \mathbf{x}_{j_t} \rangle + \frac{\rho}{2} \|\mathbf{A}_{j_t}^c \mathbf{x}_{j_t} + \sum_{j \neq k} \mathbf{A}_{j_t}^c \mathbf{x}_{j_t}^t - \mathbf{a}\|_2^2 + \eta_{j_t} B_{\phi_{j_t}}(\mathbf{x}, \mathbf{x}_{j_t}^t) \\ &= \operatorname{argmin}_{\mathbf{x}_{j_t} \in \mathcal{X}_{j_t}} f_{j_t}(\mathbf{x}_{j_t}) + \langle (\mathbf{A}_{j_t}^c)^T (\hat{\mathbf{y}}^t + \rho \mathbf{r}^t), \mathbf{x}_{j_t} \rangle + \frac{\rho}{2} \|\mathbf{A}_{j_t}^c (\mathbf{x}_{j_t} - \mathbf{x}_{j_t}^t)\|_2^2 + \eta_{j_t} B_{\phi_{j_t}}(\mathbf{x}, \mathbf{x}_{j_t}^t). \end{aligned} \quad (8)$$

Therefore, we have the algorithm of PDMM as in Algorithm 1.

To better understand PDMM, we discuss the following three aspects which play roles in choosing τ_i and ν_i : the dual backward step (7), the sparsity of \mathbf{A} and the choice of randomized blocks.

Algorithm 1 Parallel Diretion Method of Multipliers

- 1: **Input:** $\rho, \eta_j, \tau_i, \nu_i$
 - 2: **Initialization:** $\mathbf{x}^1, \hat{\mathbf{y}}^1 = \mathbf{0}$
 - 3: if τ_i, ν_i are not defined, initialize τ_i, ν_i as given in Table 1
 - 4: $\mathbf{r}^1 = \mathbf{A}\mathbf{x}^1 - \mathbf{a} = -\mathbf{a}$
 - 5: **for** $t = 1$ to T **do**
 - 6: randomly pick up j_t block coordinates
 - 7: $\mathbf{x}_{j_t}^{t+1} = \operatorname{argmin}_{\mathbf{x}_{j_t} \in \mathcal{X}_{j_t}} f_{j_t}(\mathbf{x}_{j_t}) + \langle (\mathbf{A}_{j_t}^c)^T (\hat{\mathbf{y}}^t + \rho \mathbf{r}^t), \mathbf{x}_{j_t} \rangle + \frac{\rho}{2} \|\mathbf{A}_{j_t}^c (\mathbf{x}_{j_t} - \mathbf{x}_{j_t}^t)\|_2^2 + \eta_{j_t} B_{\phi_{j_t}}(\mathbf{x}, \mathbf{x}_{j_t}^t)$
 - 8: $\mathbf{r}^{t+1} = \mathbf{r}^t + \sum_{j_t \in \mathbb{I}_t} \mathbf{A}_{j_t}^c (\mathbf{x}_{j_t}^{t+1} - \mathbf{x}_{j_t}^t)$
 - 9: $\mathbf{y}_i^{t+1} = \mathbf{y}_i^t + \tau_i \rho \mathbf{r}_i^{t+1}$
 - 10: $\hat{\mathbf{y}}_i^{t+1} = \mathbf{y}_i^{t+1} - \nu_i \rho \mathbf{r}_i^{t+1}$
 - 11: **end for**
-

Dual Backward Step: We attribute the failure of the Jacobi updates (3)-(4) to the following observation in (3), which can be rewritten as:

$$\mathbf{x}_j^{t+1} = \operatorname{argmin}_{\mathbf{x}_j \in \mathcal{X}_j} f_j(\mathbf{x}_j) + \langle \mathbf{y}^t + \rho(\mathbf{A}\mathbf{x}^t - \mathbf{a}), \mathbf{A}_j^c \mathbf{x}_j \rangle + \frac{\rho}{2} \|\mathbf{A}_j^c (\mathbf{x}_j - \mathbf{x}_j^t)\|_2^2. \quad (9)$$

In the primal \mathbf{x}_j update, the quadratic penalty term implicitly adds full gradient ascent step to the dual variable, i.e., $\mathbf{y}^t + \rho(\mathbf{A}\mathbf{x}^t - \mathbf{a})$, which we call implicit dual ascent. The implicit dual ascent along with the explicit dual ascent (4) may lead to too aggressive progress on the dual variable, particularly when the number of blocks is large. Based on this observation, we introduce an intermediate variable $\hat{\mathbf{y}}^t$ to replace \mathbf{y}^t in (9) so that the implicit dual ascent in (9) makes conservative progress, e.g., $\hat{\mathbf{y}}^t + \rho(\mathbf{A}\mathbf{x}^t - \mathbf{a}) = \mathbf{y}^t + (1 - \nu)\rho(\mathbf{A}\mathbf{x}^t - \mathbf{a})$, where $0 < \nu < 1$. $\hat{\mathbf{y}}^t$ is the result of a ‘backward step’ on the dual variable, i.e., $\hat{\mathbf{y}}^t = \mathbf{y}^t - \nu\rho(\mathbf{A}\mathbf{x}^t - \mathbf{a})$.

Moreover, one can show that τ and ν have also been implicitly used when using two-block ADMM with splitting variables (sADMM) to solve (1) [26, 38]. Section 2.2 shows sADMM is a special case of PDMM. The connection helps in understanding the role of the two parameters τ_i, ν_i in PDMM. Interestingly, the step sizes τ_i and ν_i can be improved by considering the block sparsity of \mathbf{A} and the number of random blocks K to be updated.

Sparsity of \mathbf{A} : Assume \mathbf{A} is divided into $I \times J$ blocks. While \mathbf{x}_j can be updated in parallel, the matrix multiplication $\mathbf{A}\mathbf{x}$ in the dual update (4) requires synchronization to gather messages from all block coordinates $j_t \in \mathbb{I}_t$. For updating the i -th block of the dual \mathbf{y}_i , we need $\mathbf{A}_i \mathbf{x}^{t+1} = \sum_{j_t \in \mathbb{I}_t} \mathbf{A}_{ij_t} \mathbf{x}_{j_t}^{t+1} + \sum_{k \notin \mathbb{I}_t} \mathbf{A}_{ik} \mathbf{x}_k^t$ which aggregates ‘messages’ from all \mathbf{x}_{j_t} . If \mathbf{A}_{ij_t} is a block of zeros, there is no ‘message’ from \mathbf{x}_{j_t} to \mathbf{y}_i . More precisely, $\mathbf{A}_i \mathbf{x}^{t+1} = \sum_{j_t \in \mathbb{I}_t \cap \mathcal{N}(i)} \mathbf{A}_{ij_t} \mathbf{x}_{j_t}^{t+1} + \sum_{k \notin \mathbb{I}_t} \mathbf{A}_{ik} \mathbf{x}_k^t$ where $\mathcal{N}(i)$ denotes a set of nonzero blocks in the i -th row block \mathbf{A}_i . $\mathcal{N}(i)$ can be considered as the set of neighbors of the i -th dual block \mathbf{y}_i and $d_i = |\mathcal{N}(i)|$ is the degree of the i -th dual block \mathbf{y}_i . If \mathbf{A} is sparse, d_i could be far smaller than J . According to Table 1, a low d_i will lead to bigger step sizes τ_i for the dual update and smaller step sizes for the dual backward step (7). Further, as shown in Section 2.3, when using PDMM with all blocks to solve composite minimization with overlapping blocks, PDMM can use $\tau_i = 0.5$ which is much larger than $1/J$ in sADMM.

Randomized Blocks: The number of blocks to be randomly chosen also has the effect on τ_i, ν_i . If randomly choosing one block ($K = 1$), then $\nu_i = 0, \tau_i = \frac{1}{2J-1}$. The dual backward step (7) vanishes. As K increases, ν_i increases from 0 to $1 - \frac{1}{d_i}$ and τ_i increases from $\frac{1}{2J-1}$ to $\frac{1}{d_i}$. If updating all blocks ($K = J$),

$$\tau_i = \frac{1}{d_i}, \nu_i = 1 - \frac{1}{d_i}.$$

PDMM does not necessarily choose any K combination of J blocks. The J blocks can be randomly partitioned into J/K groups where each group has K blocks. Then PDMM randomly picks one group. A simple way is to permute the J blocks and choose K blocks cyclically.

2.1 Inexact PDMM

If $\eta_{j_t} > 0$, there is an extra Bregman divergence term in (5), which can serve two purposes. First, choosing a suitable Bregman divergence can lead to a closed-form solution for (5). Second, if η_{j_t} is sufficiently large, the dual update can use a large step size ($\tau_i = 1$) and the backward step (7) can be removed ($\nu_i = 0$), leading to the same updates as PJADMM [9] (see Section 2.2).

Given a differentiable function ψ_{j_t} , its Bregman divergence is defined as

$$B_{\psi_{j_t}}(\mathbf{x}_{j_t}, \mathbf{x}_{j_t}^t) = \psi_{j_t}(\mathbf{x}_{j_t}) - \psi_{j_t}(\mathbf{x}_{j_t}^t) - \langle \nabla \psi_{j_t}(\mathbf{x}_{j_t}^t), \mathbf{x}_{j_t} - \mathbf{x}_{j_t}^t \rangle, \quad (10)$$

where $\nabla \psi_{j_t}$ denotes the gradient of ψ_{j_t} . Rearranging the terms yields

$$\psi_{j_t}(\mathbf{x}_{j_t}) - B_{\psi_{j_t}}(\mathbf{x}_{j_t}, \mathbf{x}_{j_t}^t) = \psi_{j_t}(\mathbf{x}_{j_t}^t) + \langle \nabla \psi_{j_t}(\mathbf{x}_{j_t}^t), \mathbf{x}_{j_t} - \mathbf{x}_{j_t}^t \rangle, \quad (11)$$

which is exactly the linearization of $\psi_{j_t}(\mathbf{x}_{j_t})$ at $\mathbf{x}_{j_t}^t$. Therefore, if solving (5) exactly becomes difficult due to some problematic terms, we can use the Bregman divergence to linearize these problematic terms so that (5) can be solved efficiently. More specifically, in (5), we can choose $\phi_{j_t} = \varphi_{j_t} - \frac{1}{\eta_{j_t}} \psi_{j_t}$ assuming ψ_{j_t} is the problematic term. Using the linearity of Bregman divergence,

$$B_{\phi_{j_t}}(\mathbf{x}_{j_t}, \mathbf{x}_{j_t}^t) = B_{\varphi_{j_t}}(\mathbf{x}_{j_t}, \mathbf{x}_{j_t}^t) - \frac{1}{\eta_{j_t}} B_{\psi_{j_t}}(\mathbf{x}_{j_t}, \mathbf{x}_{j_t}^t). \quad (12)$$

For instance, if f_{j_t} is a logistic function, solving (5) exactly requires an iterative algorithm. Setting $\psi_{j_t} = f_{j_t}$, $\varphi_{j_t} = \frac{1}{2} \|\cdot\|_2^2$ in (12) and plugging into (5) yield

$$\begin{aligned} \mathbf{x}_{j_t}^{t+1} &= \underset{\mathbf{x}_{j_t} \in \mathcal{X}_{j_t}}{\operatorname{argmin}} \langle \nabla f_{j_t}(\mathbf{x}_{j_t}^t), \mathbf{x}_{j_t} \rangle + \langle \hat{\mathbf{y}}^t, \mathbf{A}_{j_t} \mathbf{x}_{j_t} \rangle \\ &+ \frac{\rho}{2} \|\mathbf{A}_{j_t} \mathbf{x}_{j_t} + \sum_{k \neq j} \mathbf{A}_k \mathbf{x}_k^t - \mathbf{a}\|_2^2 + \eta_{j_t} \|\mathbf{x}_{j_t} - \mathbf{x}_{j_t}^t\|_2^2, \end{aligned} \quad (13)$$

which has a closed-form solution. Similarly, if the quadratic penalty term $\frac{\rho}{2} \|\mathbf{A}_{j_t}^c \mathbf{x}_{j_t} + \sum_{k \neq j} \mathbf{A}_k^c \mathbf{x}_k^t - \mathbf{a}\|_2^2$ is a problematic term, we can set $\psi_{j_t}(\mathbf{x}_{j_t}) = \frac{\rho}{2} \|\mathbf{A}_{j_t}^c \mathbf{x}_{j_t}\|_2^2$, then $B_{\psi_{j_t}}(\mathbf{x}_{j_t}, \mathbf{x}_{j_t}^t) = \frac{\rho}{2} \|\mathbf{A}_{j_t}^c (\mathbf{x}_{j_t} - \mathbf{x}_{j_t}^t)\|_2^2$ can be used to linearize the quadratic penalty term.

In (12), the nonnegativeness of $B_{\phi_{j_t}}$ implies that $B_{\varphi_{j_t}} \geq \frac{1}{\eta_{j_t}} B_{\psi_{j_t}}$. This condition can be satisfied as long as φ_{j_t} is more convex than ψ_{j_t} . Technically, we assume that φ_{j_t} is σ/η_{j_t} -strongly convex and ψ_{j_t} has Lipschitz continuous gradient with constant σ , which has been shown in [36]. For instance, if $\psi_{j_t}(\mathbf{x}_{j_t}) = \frac{\rho}{2} \|\mathbf{A}_{j_t}^c \mathbf{x}_{j_t}\|_2^2$, $\sigma = \rho \lambda_{\max}(\mathbf{A}_{j_t}^c)$ where $\lambda_{\max}(\mathbf{A}_{j_t}^c)$ denotes the largest eigenvalue of $(\mathbf{A}_{j_t}^c)^T \mathbf{A}_{j_t}^c$. If choosing $\varphi_{j_t} = \frac{1}{2} \|\cdot\|_2^2$, the condition is satisfied by setting $\eta_{j_t} \geq \rho \lambda_{\max}(\mathbf{A}_{j_t}^c)$.

2.2 Connections to Related Work

All blocks: There are also two other methods which update all blocks in parallel. If solving the primal updates exactly, two-block ADMM with splitting variables (sADMM) is considered in [26, 38]. We show

that sADMM is a special case of PDMM when setting $\tau_i = \frac{1}{J}$ and $\nu_i = 1 - \frac{1}{J}$ (See Appendix B). If the primal updates are solved inexactly, [9] considers a proximal Jacobian ADMM (PJADMM) by adding proximal terms where the converge rate is improved to $o(1/T)$ given the sufficiently large proximal terms. We show that PJADMM [9] is also a special case of PDMM (See Appendix C). sADMM and PJADMM are two extreme cases of PDMM. The connection between sADMM and PJADMM through PDMM can provide better understanding of the three methods and the role of dual backward step. If the primal update is solved exactly which makes sufficient progress, the dual update should take small step, e.g., sADMM. On the other hand, if the primal update takes small progress by adding proximal terms, the dual update can take full gradient step, e.g. PJADMM. While sADMM is a direct derivation of ADMM, PJADMM introduces more terms and parameters.

Randomized blocks: While PDMM can randomly update any number of blocks, RBUSMM [19] can only randomly update one block. The convergence of RBSUMM requires certain local error bounds to be hold and decreasing step size. Moreover, the iteration complexity of RBSUMM is still unknown. In contrast, PDMM converges at a rate of $O(1/T)$ with the constant step size.

2.3 Randomized Overlapping Block Coordinate

Consider the composite minimization problem of a sum of a loss function $\ell(\mathbf{w})$ and composite regularizers $g_j(\mathbf{w}_j)$:

$$\min_{\mathbf{w}} \ell(\mathbf{w}) + \sum_{j=1}^L g_j(\mathbf{w}_j), \quad (14)$$

which considers L overlapping groups $\mathbf{w}_j \in \mathbb{R}^{b \times 1}$. Let $J = L + 1$, $\mathbf{x}_J = \mathbf{w}$. For $1 \leq j \leq L$, denote $\mathbf{x}_j = \mathbf{w}_j$, then $\mathbf{x}_j = \mathbf{U}_j^T \mathbf{x}_J$, where $\mathbf{U}_j \in \mathbb{R}^{b \times L}$ is the columns of an identity matrix and extracts the coordinates of \mathbf{x}_j . Denote $\mathbf{U} = [\mathbf{U}_1, \dots, \mathbf{U}_L] \in \mathbb{R}^{n \times (bL)}$ and $\mathbf{A} = [\mathbf{I}_{bL}, -\mathbf{U}^T]$ where bL denotes $b \times L$. By letting $f_j(\mathbf{x}_j) = g_j(\mathbf{w}_j)$ and $f_J(\mathbf{x}_J) = \ell(\mathbf{w})$, (14) can be written as:

$$\min_{\mathbf{x}} \sum_{j=1}^J f_j(\mathbf{x}_j) \quad \text{s.t.} \quad \mathbf{A}\mathbf{x} = \mathbf{0}. \quad (15)$$

where $\mathbf{x} = [\mathbf{x}_1; \dots; \mathbf{x}_L; \mathbf{x}_{L+1}] \in \mathbb{R}^{b \times J}$. (15) can be solved by PDMM in a randomized block coordinate fashion. In \mathbf{A} , for b rows block, there are only two nonzero blocks, i.e., $d_i = 2$. Therefore, $\tau_i = \frac{K}{2(2J-K)}, \nu_i = 0.5$. In particular, if $K = J$, $\tau_i = \nu_i = 0.5$. In contrast, sADMM uses $\tau_i = 1/J \ll 0.5, \nu_i = 1 - 1/J > 0.5$ if J is larger.

Remark 1 (a) ADMM [3] can solve (15) where the equality constraint is $\mathbf{x}_j = \mathbf{U}_j^T \mathbf{x}_J$.

(b) In this setting, Gauss-Seidel ADMM (GSADMM) and BSUMM [19] are the same as ADMM. BSUMM should converge with constant stepsize ρ (not necessarily sufficiently small), although the theory of BSUMM does not include this special case.

(c) Consensus optimization [3] has the same formulation as (15). Therefore, PDMM can also be used as a randomized consensus optimization algorithm.

3 Theoretical Results

We establish the convergence results for PDMM under fairly simple assumptions:

Assumption 1

- (1) $f_j : \mathbb{R}^{n_j} \rightarrow \mathbb{R} \cup \{+\infty\}$ are closed, proper, and convex.
(2) A KKT point of the Lagrangian ($\rho = 0$ in (2)) of Problem (1) exists.

Assumption 2 is the same as that required by ADMM [3, 35]. Assume that $\{\mathbf{x}_j^*, \mathbf{y}_i^*\}$ satisfies the KKT conditions of the Lagrangian ($\rho = 0$ in (2)), i.e.,

$$-\mathbf{A}_j^T \mathbf{y}^* \in \partial f_j(\mathbf{x}_j^*), \quad (16)$$

$$\mathbf{A}\mathbf{x}^* - \mathbf{a} = 0. \quad (17)$$

During iterations, (82) is satisfied if $\mathbf{A}\mathbf{x}^{t+1} = \mathbf{a}$. Let ∂f_j be the subdifferential of f_j . The optimality conditions for the \mathbf{x}_j update (5) is

$$-\mathbf{A}_j^c[\mathbf{y}^t + (1-\nu)\rho(\mathbf{A}\mathbf{x}^t - \mathbf{a}) + \mathbf{A}_j^c(\mathbf{x}_j^{t+1} - \mathbf{x}_j^t)] - \eta_j(\nabla\phi_j(\mathbf{x}_j^{t+1}) - \nabla\phi_j(\mathbf{x}_j^t)) \in \partial f_j(\mathbf{x}_j^{t+1}). \quad (18)$$

When $\mathbf{A}\mathbf{x}^{t+1} = \mathbf{a}$, $\mathbf{y}^{t+1} = \mathbf{y}^t$. If $\mathbf{A}_j^c(\mathbf{x}_j^{t+1} - \mathbf{x}_j^t) = 0$, then $\mathbf{A}\mathbf{x}^t - \mathbf{a} = 0$. When $\eta_j \geq 0$, further assuming $B_{\phi_j}(\mathbf{x}_j^{t+1}, \mathbf{x}_j^t) = 0$, (81) will be satisfied. Overall, the KKT conditions (81)-(82) are satisfied if the following optimality conditions are satisfied by the iterates:

$$\mathbf{A}\mathbf{x}^{t+1} = \mathbf{a}, \mathbf{A}_j^c(\mathbf{x}_j^{t+1} - \mathbf{x}_j^t) = 0, \quad (19)$$

$$B_{\phi_j}(\mathbf{x}_j^{t+1}, \mathbf{x}_j^t) = 0. \quad (20)$$

The above optimality conditions are sufficient for the KKT conditions. (85) are the optimality conditions for the exact PDMM. (86) is needed only when $\eta_j > 0$.

Let $\mathbf{z}_{ij} = \mathbf{A}_{ij}\mathbf{x}_j \in \mathbb{R}^{m_i \times 1}$, $\mathbf{z}_i^r = [\mathbf{z}_{i1}^T, \dots, \mathbf{z}_{iJ}^T]^T \in \mathbb{R}^{m_i J \times 1}$ and $\mathbf{z} = [(\mathbf{z}_1^r)^T, \dots, (\mathbf{z}_I^r)^T]^T \in \mathbb{R}^{Jm \times 1}$. Define the residual of optimality conditions (85)-(86) as

$$R(\mathbf{x}^{t+1}) = \frac{\rho}{2} \|\mathbf{z}^{t+1} - \mathbf{z}^t\|_{\mathbf{P}_t}^2 + \frac{\rho}{2} \sum_{i=1}^I \beta_i \|\mathbf{A}_i^r \mathbf{x}^{t+1} - \mathbf{a}_i\|_2^2 + \sum_{j=1}^J \eta_j B_{\phi_j}(\mathbf{x}_j^{t+1}, \mathbf{x}_j^t). \quad (21)$$

where \mathbf{P}_t is some positive semi-definite matrix¹ and $\beta_i = \frac{K}{JK_i}$. If $R(\mathbf{x}^{t+1}) \rightarrow 0$, (85)-(86) will be satisfied and thus PDMM converges to the KKT point $\{\mathbf{x}^*, \mathbf{y}^*\}$. Define the current iterate $\mathbf{v}^t = (\mathbf{x}_j^t, \mathbf{y}_i^t)$ and $h(\mathbf{v}^*, \mathbf{v}^t)$ as a distance from \mathbf{v}^t to a KKT point $\mathbf{v}^* = (\mathbf{x}_j^*, \mathbf{y}_i^*)$:

$$h(\mathbf{v}^*, \mathbf{v}^t) = \frac{K}{J} \sum_{i=1}^I \frac{1}{2\tau_i \rho} \|\mathbf{y}_i^* - \mathbf{y}_i^{t-1}\|_2^2 + \tilde{\mathcal{L}}_\rho(\mathbf{x}^t, \mathbf{y}^t) + \frac{\rho}{2} \|\mathbf{z}^* - \mathbf{z}^t\|_{\mathbf{Q}}^2 + \sum_{j=1}^J \eta_j B_{\phi_j}(\mathbf{x}_j^*, \mathbf{x}_j^t), \quad (22)$$

where \mathbf{Q} is a positive semi-definite matrix¹ and $\tilde{\mathcal{L}}_\rho(\mathbf{x}^t, \mathbf{y}^t)$ with $\gamma_i = \frac{2(J-K)}{K_i(2J-K)} + \frac{1}{d_i} - \frac{K}{JK_i}$ is

$$\tilde{\mathcal{L}}_\rho(\mathbf{x}^t, \mathbf{y}^t) = f(\mathbf{x}^t) - f(\mathbf{x}^*) + \sum_{i=1}^I \left\{ \langle \mathbf{y}_i^t, \mathbf{A}_i^r \mathbf{x}^t - \mathbf{a}_i \rangle + \frac{(\gamma_i - \tau_i)\rho}{2} \|\mathbf{A}_i^r \mathbf{x}^t - \mathbf{a}_i\|_2^2 \right\}. \quad (23)$$

The following Lemma shows that $h(\mathbf{v}^*, \mathbf{v}^t) \geq 0$.

¹See the definition in the Appendix A.

Lemma 1 Let $\mathbf{v}^t = (\mathbf{x}_j^t, \mathbf{y}_i^t)$ be generated by PDMM (5)-(7) and $h(\mathbf{v}^*, \mathbf{v}^t)$ be defined in (89). Setting $\nu_i = 1 - \frac{1}{K_i}$ and $\tau_i = \frac{K}{K_i(2J-K)}$, we have

$$h(\mathbf{v}^*, \mathbf{v}^t) \geq \frac{\rho}{2} \sum_{i=1}^I \zeta_i \|\mathbf{A}_i^r \mathbf{x}^t - \mathbf{a}_i\|_2^2 + \frac{\rho}{2} \|\mathbf{z}^* - \mathbf{z}^t\|_{\mathbf{Q}}^2 + \sum_{j=1}^J \eta_j B_{\phi_j}(\mathbf{x}_j^*, \mathbf{x}_j^t) \geq 0. \quad (24)$$

where $\zeta_i = \frac{J-K}{K_i(2J-K)} + \frac{1}{d_i} - \frac{K}{JK_i} \geq 0$. Moreover, if $h(\mathbf{v}^*, \mathbf{v}^t) = 0$, then $\mathbf{A}_i^r \mathbf{x}^t = \mathbf{a}_i$, $\mathbf{z}^t = \mathbf{z}^*$ and $B_{\phi_j}(\mathbf{x}_j^*, \mathbf{x}_j^t) = 0$. Thus, (16)-(17) are satisfied.

In PDMM, \mathbf{y}^{t+1} depends on \mathbf{x}^{t+1} , which in turn depends on \mathbb{I}_t . \mathbf{x}^t and \mathbf{y}^t are independent of \mathbb{I}_t . \mathbf{x}^t depends on the observed realizations of the random variable

$$\xi_{t-1} = \{\mathbb{I}_1, \dots, \mathbb{I}_{t-1}\}. \quad (25)$$

The following theorem shows that $h(\mathbf{v}^*, \mathbf{v}^t)$ decreases monotonically and thus establishes the global convergence of PDMM.

Theorem 1 (Global Convergence of PDMM) Let $\mathbf{v}^t = (\mathbf{x}_j^t, \mathbf{y}_i^t)$ be generated by PDMM (5)-(7) and $\mathbf{v}^* = (\mathbf{x}_j^*, \mathbf{y}_i^*)$ be a KKT point satisfying (16)-(17). Setting $\nu_i = 1 - \frac{1}{K_i}$ and $\tau_i = \frac{K}{K_i(2J-K)}$, we have

$$0 \leq \mathbb{E}_{\xi_t} h(\mathbf{v}^*, \mathbf{v}^{t+1}) \leq \mathbb{E}_{\xi_{t-1}} h(\mathbf{v}^*, \mathbf{v}^t), \quad \mathbb{E}_{\xi_t} R(\mathbf{x}^{t+1}) \rightarrow 0. \quad (26)$$

The following theorem establishes the iteration complexity of PDMM in an ergodic sense.

Theorem 2 (The Rate of Convergence) Let $(\mathbf{x}_j^t, \mathbf{y}_i^t)$ be generated by PDMM (5)-(7). Let $\bar{\mathbf{x}}^T = \sum_{t=1}^T \mathbf{x}^t$. Setting $\nu_i = 1 - \frac{1}{K_i}$ and $\tau_i = \frac{K}{K_i(2J-K)}$, we have

$$\mathbb{E} f(\bar{\mathbf{x}}^T) - f(\mathbf{x}^*) \leq \frac{\frac{J}{K} \left\{ \sum_{i=1}^I \frac{1}{2\beta_i \rho} \|\mathbf{y}_i^*\|_2^2 + \tilde{\mathcal{L}}_{\rho}(\mathbf{x}^1, \mathbf{y}^1) + \frac{\rho}{2} \|\mathbf{z}^* - \mathbf{z}^1\|_{\mathbf{Q}}^2 + \sum_{j=1}^J \eta_j B_{\phi_j}(\mathbf{x}_j^*, \mathbf{x}_j^1) \right\}}{T}, \quad (27)$$

$$\mathbb{E} \sum_{i=1}^I \beta_i \|\mathbf{A}_i^r \bar{\mathbf{x}}^T - \mathbf{a}_i\|_2^2 \leq \frac{2}{\rho} \frac{h(\mathbf{v}^*, \mathbf{v}^0)}{T}. \quad (28)$$

where $\beta_i = \frac{K}{JK_i}$ and \mathbf{Q} is a positive semi-definite matrix.

4 Extensions: PDMM with Randomized Dual Block Coordinate Ascent

In this section, we further show that PDMM can update the dual blocks randomly. The randomized dual block coordinate ascent (RDBCD) can further increase of dual step size τ_i . More specifically, at time $t+1$, PDMM randomly selects K primal blocks denoted by \mathbb{J}_t and K_I dual blocks denoted by set \mathbb{I}_t , then executes the following iterates:

$$\hat{\mathbf{y}}_i^{t+1} = \mathbf{y}_i^{t+1} - \nu_i \rho (\mathbf{A}_i \mathbf{x}^{t+1} - \mathbf{a}_i), \quad (29)$$

$$\mathbf{x}_{j_t}^{t+1} = \operatorname{argmin}_{\mathbf{x}_{j_t} \in \mathcal{X}_{j_t}} L_{\rho}(\mathbf{x}_{j_t}, \mathbf{x}_{k \neq j_t}^t, \hat{\mathbf{y}}^t) + \eta_{j_t} B_{\phi_{j_t}}(\mathbf{x}_{j_t}, \mathbf{x}_{j_t}^t), \quad j_t \in \mathbb{J}_t, \quad (30)$$

$$\mathbf{y}_{i_t}^{t+1} = \mathbf{y}_{i_t}^t + \tau_{i_t} \rho (\mathbf{A}_{i_t} \mathbf{x}^{t+1} - \mathbf{a}_{i_t}), \quad i_t \in \mathbb{I}_t, \quad (31)$$

where $\mathbf{x} = (\mathbf{x}_{j_t \in \mathbb{J}_t}^{t+1}, \mathbf{x}_{k \notin \mathbb{J}_t})$, $\mathbf{y} = (\mathbf{y}_{i_t \in \mathbb{I}_t}^{t+1}, \mathbf{y}_{k \notin \mathbb{I}_t}^t)$, and τ_i, ν_i take the following values:

$$\tau_i = \frac{K}{\tilde{K}_i[(2J-K)\frac{K_I}{T} + K(1-\frac{K_I}{T})]} \geq \frac{K}{\tilde{K}_i(2J-K)}, \nu_i = 1 - \frac{1}{\tilde{K}_i}. \quad (32)$$

The dual step size τ_i increases when using RDBCD. If $I = J, K_I = K = 1$, $\tau_i = \frac{I}{3J-2} > \frac{1}{3}$, which is far greater than $\frac{1}{2J-1}$ in PDMM without RDBCD.

In this setting, \mathbf{x}^{t+1} depends on \mathbb{J}_t , and \mathbf{y}^{t+1} depends on $\mathbb{I}_t, \mathbb{J}_t$. \mathbf{y}^{t+1} depends on \mathbf{x}^{t+1} , which in turn depends on the observed realizations of the random variable

$$\xi_t = \{(\mathbb{I}_1, \mathbb{J}_1), \dots, (\mathbb{I}_t, \mathbb{J}_t)\}. \quad (33)$$

Define the current iterate $\mathbf{v}^t = (\mathbf{x}_j^t, \mathbf{y}_i^t)$ and $h(\mathbf{v}^*, \mathbf{v}^t)$ as a distance from \mathbf{v}^t to a KKT point $\mathbf{v}^* = (\mathbf{x}_j^*, \mathbf{y}_i^*)$:

$$h(\mathbf{v}^*, \mathbf{v}^t) = \frac{K}{J} \sum_{i=1}^I \frac{I}{2K_I \tau_i \rho} \|\mathbf{y}_i^* - \mathbf{y}_i^{t-1}\|_2^2 + \tilde{\mathcal{L}}_\rho(\mathbf{x}^t, \mathbf{y}^t) + \frac{\rho}{2} \|\mathbf{z}^* - \mathbf{z}^t\|_{\mathbf{Q}}^2 + \boldsymbol{\eta}^T B_\phi(\mathbf{x}^*, \mathbf{x}^t). \quad (34)$$

The following Lemma shows that $h(\mathbf{v}^*, \mathbf{v}^t) \geq 0$.

Lemma 2 Let $h(\mathbf{v}^*, \mathbf{v}^t)$ be defined in (34). Setting $\nu_i = 1 - \frac{1}{\tilde{K}_i}$ and $\tau_i = \frac{K}{\tilde{K}_i[(2J-K)\frac{K_I}{T} + K(1-\frac{K_I}{T})]}$, we have

$$h(\mathbf{v}^*, \mathbf{v}^t) \geq \frac{\rho}{2} \sum_{i=1}^I \zeta_i \|\mathbf{A}_i^r \mathbf{x}^t - \mathbf{a}_i\|_2^2 + \frac{\rho}{2} \|\mathbf{z}^* - \mathbf{z}^t\|_{\mathbf{Q}}^2 + \sum_{j=1}^J \eta_j B_{\phi_j}(\mathbf{x}_j^*, \mathbf{x}_j^t) \geq 0. \quad (35)$$

where $\zeta_i = \frac{(J-K)\frac{K_I}{T} + K(1-\frac{K_I}{T})}{\tilde{K}_i[(2J-K)\frac{K_I}{T} + K(1-\frac{K_I}{T})]} + \frac{1}{d_i} - \frac{K}{JK_i} \geq 0$. Moreover, if $h(\mathbf{v}^*, \mathbf{v}^t) = 0$, then $\mathbf{A}_i^r \mathbf{x}^t = \mathbf{a}_i$, $\mathbf{z}^t = \mathbf{z}^*$ and $B_{\phi_j}(\mathbf{x}_j^*, \mathbf{x}_j^t) = 0$. Thus, (16)-(17) are satisfied.

The following theorem shows that $h(\mathbf{v}^*, \mathbf{v}^t)$ decreases monotonically and thus establishes the global convergence of PDMM.

Theorem 3 (Global Convergence) Let $\mathbf{v}^t = (\mathbf{x}_{j_t}^t, \mathbf{y}_{i_t}^t)$ be generated by PDMM (29)-(31) and $\mathbf{v}^* = (\mathbf{x}_j^*, \mathbf{y}_i^*)$ be a KKT point satisfying (16)-(17). Setting $\nu_i = 1 - \frac{1}{\tilde{K}_i}$ and $\tau_i = \frac{K}{\tilde{K}_i[(2J-K)\frac{K_I}{T} + K(1-\frac{K_I}{T})]}$, we have

$$0 \leq \mathbb{E}_{\xi_t} h(\mathbf{v}^*, \mathbf{v}^{t+1}) \leq \mathbb{E}_{\xi_{t-1}} h(\mathbf{v}^*, \mathbf{v}^t), \quad \mathbb{E}_{\xi_t} R(\mathbf{x}^{t+1}) \rightarrow 0. \quad (36)$$

Theorem 4 (The Rate of Convergence) Let $(\mathbf{x}_{j_t}^t, \mathbf{y}_{i_t}^t)$ be generated by PDMM (29)-(31). Let $\bar{\mathbf{x}}^T = \sum_{t=1}^T \mathbf{x}^t$. Setting $\nu_i = 1 - \frac{1}{\tilde{K}_i}$ and $\tau_i = \frac{K}{\tilde{K}_i[(2J-K)\frac{K_I}{T} + K(1-\frac{K_I}{T})]}$, we have

$$\mathbb{E} f(\bar{\mathbf{x}}^T) - f(\mathbf{x}^*) \leq \frac{\frac{I}{K_I} \sum_{i=1}^I \frac{1}{2\tau_i \rho} \|\mathbf{y}_i^0\|_2^2 + \frac{J}{K} \left\{ \frac{1}{2\beta_i \rho} \|\mathbf{y}_i^*\|_2^2 + \tilde{\mathcal{L}}_\rho(\mathbf{x}^1, \mathbf{y}^1) + \frac{\rho}{2} \|\mathbf{z}^* - \mathbf{z}^1\|_{\mathbf{Q}}^2 + \boldsymbol{\eta}^T B_\phi(\mathbf{x}^*, \mathbf{x}^1) \right\}}{T}, \quad (37)$$

$$\mathbb{E} \sum_{i=1}^I \beta_i \|\mathbf{A}_i^r \bar{\mathbf{x}}^T - \mathbf{a}_i\|_2^2 \leq \frac{\frac{2}{\rho} h(\mathbf{v}^*, \mathbf{v}^0)}{T}. \quad (38)$$

where $\beta_i = \frac{K}{JK_i}$.

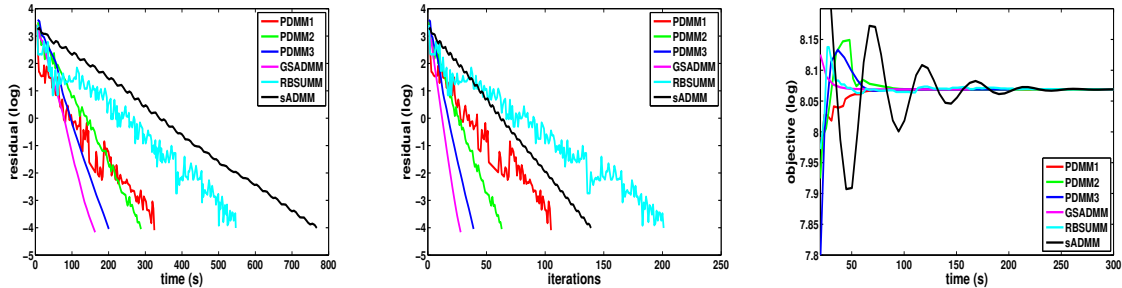


Figure 1: Comparison of the convergence of PDMM (with K blocks) with ADMM methods in RPCA. The values of τ_i, ν_i in PDMM is computed according to Table 1. Gauss-Seidel (GSADMM) is the fastest algorithm, although whether it converges or not is unknown. PDMM3 is faster than PDMM1 and PDMM2. For the two randomized one block coordinate methods, PDMM1 is faster than RBSUMM.

Table 2: The 'best' results of PDMM with tuning parameters τ_i, ν_i in RPCA. PDMM1 randomly updates one block and is the fastest algorithm. PDMMs converges faster than other ADMM methods.

	time (s)	iteration	residual($\times 10^{-5}$)	objective (log)
PDMM1	118.83	40	3.60	8.07
PDMM2	137.46	34	5.51	8.07
PDMM3	147.82	31	6.54	8.07
GSADMM	163.09	28	6.84	8.07
RBSUMM	206.96	141	8.55	8.07
sADMM ²	731.51	139	9.73	8.07

5 Experimental Results

In this section, we evaluate the performance of PDMM in solving robust principal component analysis (RPCA) and overlapping group lasso [42]. We compared PDMM with ADMM [3] or GSADMM (no theory guarantee), sADMM [26, 38], and RBSUMM [19]. Note GSADMM includes BSUMM [19]. All experiments are implemented in Matlab and run sequentially. We run the experiments 10 times and report the average results. The stopping criterion is either residual $\frac{\text{norm}(x\text{-yold})}{\text{norm}(x\text{old})} + \frac{\text{norm}(y\text{-yold})}{\text{norm}(y\text{old})} \leq 10^{-4}$ or the maximum number of iterations.

RPCA: RPCA is used to obtain a low rank and sparse decomposition of a given matrix \mathbf{A} corrupted by noise [5, 26]:

$$\min \frac{1}{2} \|\mathbf{X}_1\|_F^2 + \gamma_2 \|\mathbf{X}_2\|_1 + \gamma_3 \|\mathbf{X}_3\|_* \text{ s.t. } \mathbf{A} = \mathbf{X}_1 + \mathbf{X}_2 + \mathbf{X}_3. \quad (39)$$

where $\mathbf{A} \in \mathbb{R}^{m \times n}$, \mathbf{X}_1 is a noise matrix, \mathbf{X}_2 is a sparse matrix and \mathbf{X}_3 is a low rank matrix. $\mathbf{A} = \mathbf{L} + \mathbf{S} + \mathbf{V}$ is generated in the same way as [26]². In this experiment, $m = 1000, n = 5000$ and the rank is 100. The number appended to PDMM denotes the number of blocks (K) to be chosen in PDMM, e.g., PDMM1 randomly updates one block.

Figure 1 compares the convergence results of PDMM with ADMM methods. In PDMM, $\rho = 1$ and τ_i, ν_i are chosen according to Table (1), i.e., $(\tau_i, \nu_i) = \{(\frac{1}{5}, 0), (\frac{1}{4}, \frac{1}{2}), (\frac{1}{3}, \frac{1}{3})\}$ for PDMM1, PDMM2 and PDMM3

²http://www.stanford.edu/~boyd/papers/prox_algs/matrix_decomp.html

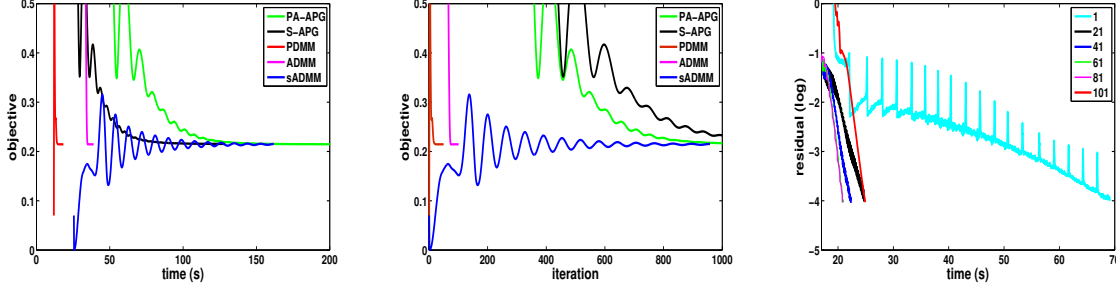


Figure 2: Comparison of convergence of PDMM and other methods in overlapping group Lasso.

respectively. We choose the 'best' results for GSADMM ($\rho = 1$) and RBSUMM ($\rho = 1, \alpha = \rho \frac{11}{\sqrt{t}+10}$) and sADMM ($\rho = 1$). PDMMs perform better than RBSUMM and sADMM. Note the public available code of sADMM² does not have dual update, i.e., $\tau_i = 0$. sADMM should be the same as PDMM3 if $\tau_i = \frac{1}{3}$. Since $\tau_i = 0$, sADMM is the slowest algorithm. Without tuning the parameters of PDMM, GSADMM converges faster than PDMM. Note PDMM can run in parallel but GSADMM only runs sequentially. PDMM3 is faster than two randomized version of PDMM since the costs of extra iterations in PDMM1 and PDMM2 have surpassed the savings at each iteration. For the two randomized one block coordinate methods, PDMM1 converges faster than RBSUMM in terms of both the number of iterations and runtime.

The effect of τ_i, ν_i : We tuned the parameter τ_i, ν_i in PDMMs. Three randomized methods (RBSUMM, PDMM1 and PDMM2) choose the blocks cyclically instead of randomly. Table 2 compares the 'best' results of PDMM with other ADMM methods. In PDMM, $(\tau_i, \nu_i) = \{(\frac{1}{2}, 0), (\frac{1}{3}, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2})\}$. GSADMM converges with the smallest number of iterations, but PDMMs can converge faster than GSADMM in terms of runtime. Since GSADMM uses new iterates which increases computation compared to PDMM3, PDMM3 can be faster than GSADMM if the numbers of iterations are close. PDMM1 and PDMM2 can be faster than PDMM3. By simply updating one block, PDMM1 is the fastest algorithm and achieves the lowest residual.

Overlapping Group Lasso: We consider solving the overlapping group lasso problem [42]:

$$\min_{\mathbf{w}} \frac{1}{2L\lambda} \|\mathbf{A}\mathbf{w} - \mathbf{b}\|_2^2 + \sum_{g \in \mathcal{G}} d_g \|\mathbf{w}_g\|_2. \quad (40)$$

where $\mathbf{A} \in \mathbb{R}^{m \times n}$, $\mathbf{w} \in \mathbb{R}^{n \times 1}$ and $\mathbf{w}_g \in \mathbb{R}^{b \times 1}$ is the vector of overlapping group indexed by g . d_g is some positive weight of group $g \in \mathcal{G}$. As shown in Section 2.3, (40) can be rewritten as the form (15). The data is generated in a same way as [41, 8]: the elements of \mathbf{A} are sampled from normal distribution, $b = \mathbf{A}\mathbf{x} + \epsilon$ with noise ϵ sampled from normal distribution, and $\mathbf{x}_j = (-1)^j \exp(-(j-1)/100)$. In this experiment, $m = 5000$, the number of groups is $L = 100$, and $d_g = \frac{1}{L}, \lambda = \frac{L}{5}$ in (40). The size of each group is 100 and the overlap is 10. The total number of blocks in PDMM and sADMM is $J = 101$. τ_i, ν_i in PDMM are computed according to Table (1).

In Figure 2, the first two figures plot the convergence of objective in terms of the number of iterations and time. PDMM uses all 101 blocks and is the fastest algorithm. ADMM is the same as GSADMM in this problem, but is slower than PDMM. Since sADMM does not consider the sparsity, it uses $\tau_i = \frac{1}{J+1}, \nu_i = 1 - \frac{1}{J+1}$, leading to slow convergence. The two accelerated methods, PA-APG [41] and S-APG [8], are slower than PDMM and ADMM.

The effect of K : The third figure shows PDMM with different number of blocks K . Although the complexity of each iteration is the lowest when $K = 1$, PDMM takes much more iterations than other

cases and thus takes the longest time. As K increases, PDMM converges faster and faster. When $K = 20$, the runtime is already same as using all blocks. When $K > 21$, PDMM takes less time to converge than using all blocks. The runtime of PDMM decreases as K increases from 21 to 61. However, the speedup from 61 to 81 is negligible. We tried different set of parameters for RBSUMM $\rho^{\frac{i^2+1}{i+t}}$ ($0 \leq i \leq 5, \rho = 0.01, 0.1, 1$) or sufficiently small step size, but did not see the convergence of the objective within 5000 iterations. Therefore, the results are not included here.

6 Conclusions

We proposed a randomized block coordinate variant of ADMM named Parallel Direction Method of Multipliers (PDMM) to solve the class of problem of minimizing block-separable convex functions subject to linear constraints. PDMM considers the sparsity and the number of blocks to be updated when setting the step size. We show two other Jacobian ADMM methods are two special cases of PDMM. We also use PDMM to solve overlapping block problems. The global convergence and the iteration complexity are established with constant step size. Experiments on robust PCA and overlapping group lasso show that PDMM is faster than existing methods.

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A Convergence Analysis

A.1 Technical Preliminaries

We first define some notations will be used specifically in this section. Let $\mathbf{z}_{ij} = \mathbf{A}_{ij}\mathbf{x}_j \in \mathbb{R}^{m_i \times 1}$, $\mathbf{z}_i^r = [\mathbf{z}_{i1}^T, \dots, \mathbf{z}_{iJ}^T]^T \in \mathbb{R}^{m_i J \times 1}$ and $\mathbf{z} = [(\mathbf{z}_1^r)^T, \dots, (\mathbf{z}_I^r)^T]^T \in \mathbb{R}^{Jm \times 1}$. Let $\mathbf{W}_i \in \mathbb{R}^{Jm_i \times m_i}$ be a column vector of $\mathbf{W}_{ij} \in \mathbb{R}^{m_i \times m_i}$ where

$$\mathbf{W}_{ij} = \begin{cases} \mathbf{I}_{m_i}, & \text{if } \mathbf{A}_{ij} \neq \mathbf{0}, \\ \mathbf{0} & \text{otherwise.} \end{cases} \quad (41)$$

Define $\mathbf{Q} \in \mathbb{R}^{Jm \times Jm}$ as a diagonal matrix of $\mathbf{Q}_i \in \mathbb{R}^{Jm_i \times Jm_i}$ and

$$\mathbf{Q} = \text{diag}([\mathbf{Q}_1, \dots, \mathbf{Q}_I]), \mathbf{Q}_i = \text{diag}(\mathbf{W}_i) - \frac{1}{d_i} \mathbf{W}_i \mathbf{W}_i^T. \quad (42)$$

Therefore, for an optimal solution \mathbf{x}^* satisfying $\mathbf{A}\mathbf{x}^* = \mathbf{a}$, we have

$$\begin{aligned} \|\mathbf{z}^t - \mathbf{z}^*\|_{\mathbf{Q}}^2 &= \sum_{i=1}^I \|\mathbf{z}_i^t - \mathbf{z}_i^*\|_{\mathbf{Q}_i}^2 = \sum_{i=1}^I \|\mathbf{z}_i^t - \mathbf{z}_i^*\|_{\text{diag}(\mathbf{w}_i) - \frac{1}{d_i} \mathbf{w}_i \mathbf{w}_i^T}^2 \\ &= \sum_{i=1}^I \left[\sum_{j \in \mathcal{N}(i)} \|\mathbf{z}_{ij}^t - \mathbf{z}_{ij}^*\|_2^2 - \frac{1}{d_i} \|\mathbf{w}_i^T (\mathbf{z}_i^t - \mathbf{z}_i^*)\|_2^2 \right] \\ &= \sum_{i=1}^I \left[\|\mathbf{z}_i^t - \mathbf{z}_i^*\|_2^2 - \frac{1}{d_i} \|\mathbf{A}_i^r \mathbf{x}^t - \mathbf{a}_i\|_2^2 \right], \end{aligned} \quad (43)$$

where the last equality uses $\mathbf{w}_i^T \mathbf{z}_i^* = \mathbf{A}_i^r \mathbf{x}^* = \mathbf{a}_i$.

In the following lemma, we prove that \mathbf{Q}_i is a positive semi-definite matrix. Thus, \mathbf{Q} is also positive semi-definite.

Lemma 3 \mathbf{Q}_i is positive semi-definite.

Proof: As \mathbf{W}_{ij} is either an identity matrix or a zero matrix, \mathbf{W}_i has d_i nonzero entries. Removing the zero entries from \mathbf{W}_i , we have $\tilde{\mathbf{W}}_i$ which only has d_i nonzero entries. Then,

$$\tilde{\mathbf{W}}_i = \begin{bmatrix} \mathbf{I}_{m_i} \\ \vdots \\ \mathbf{I}_{m_i} \end{bmatrix}, \text{diag}(\tilde{\mathbf{W}}_i) = \begin{bmatrix} \mathbf{I}_{m_i} & & \\ & \ddots & \\ & & \mathbf{I}_{m_i} \end{bmatrix}, \quad (44)$$

$\text{diag}(\mathbf{W}_i)$ is an identity matrix. Define $\tilde{\mathbf{Q}}_i = \text{diag}(\tilde{\mathbf{W}}_i) - \frac{1}{d_i} \tilde{\mathbf{W}}_i \tilde{\mathbf{W}}_i^T$. If $\tilde{\mathbf{Q}}_i$ is positive semi-definite, \mathbf{Q}_i is positive semi-definite.

Denote $\lambda_{\tilde{\mathbf{W}}_i}^{\max}$ as the largest eigenvalue of $\tilde{\mathbf{W}}_i \tilde{\mathbf{W}}_i^T$, which is equivalent to the largest eigenvalue of $\tilde{\mathbf{W}}_i^T \tilde{\mathbf{W}}_i$. Since $\tilde{\mathbf{W}}_i^T \tilde{\mathbf{W}}_i = d_i \mathbf{I}_{m_i}$, then $\lambda_{\tilde{\mathbf{W}}_i}^{\max} = d_i$. Then, for any \mathbf{v} ,

$$\|\mathbf{v}\|_{\tilde{\mathbf{W}}_i \tilde{\mathbf{W}}_i^T}^2 \leq \lambda_{\tilde{\mathbf{W}}_i}^{\max} \|\mathbf{v}\|_2^2 = d_i \|\mathbf{v}\|_2^2. \quad (45)$$

Thus,

$$\|\mathbf{v}\|_{\mathbf{Q}_i}^2 = \|\mathbf{v}\|_{\text{diag}(\tilde{\mathbf{W}}_i) - \frac{1}{d_i} \tilde{\mathbf{W}}_i \tilde{\mathbf{W}}_i^T}^2 = \|\mathbf{v}\|_2^2 - \frac{1}{d_i} \|\mathbf{v}\|_{\tilde{\mathbf{W}}_i \tilde{\mathbf{W}}_i^T}^2 \geq 0, \quad (46)$$

which completes the proof. \blacksquare

Let $\mathbf{W}_i^t \in \mathbb{R}^{Jm_i \times m_i}$ be a column vector of $\mathbf{W}_{ijt} \in \mathbb{R}^{m_i \times m_i}$ where

$$\mathbf{W}_{ijt} = \begin{cases} \mathbf{I}_{m_i}, & \text{if } \mathbf{A}_{ijt} \neq \mathbf{0} \text{ and } jt \in \mathbb{I}_t, \\ \mathbf{0} & \text{otherwise.} \end{cases} \quad (47)$$

Define $\mathbf{P}_t \in \mathbb{R}^{Jm \times Jm}$ as a diagonal matrix of $\mathbf{P}_i^t \in \mathbb{R}^{Jm_i \times Jm_i}$ and

$$\mathbf{P}_t = \text{diag}[\mathbf{P}_1^t, \dots, \mathbf{P}_I^t], \mathbf{P}_i^t = \text{diag}(\mathbf{W}_i^t) - \frac{1}{\tilde{K}_i} \mathbf{W}_i^t (\mathbf{W}_i^t)^T. \quad (48)$$

where $\tilde{K}_i = \min\{K, d_i\} \geq \min\{|\mathbb{I}_t \cap \mathcal{N}_i|, d_i\}$. Using similar arguments in Lemma 3, we can show \mathbf{P}_t is positive semi-definite. Therefore,

$$\begin{aligned} \|\mathbf{z}^{t+1} - \mathbf{z}^t\|_{\mathbf{P}_t}^2 &= \sum_{i=1}^I \|\mathbf{z}_i^{t+1} - \mathbf{z}_i^t\|_{\mathbf{P}_i^t}^2 = \sum_{i=1}^I \|\mathbf{z}_i^{t+1} - \mathbf{z}_i^t\|_{\text{diag}(\mathbf{w}_i^t) - \frac{1}{\tilde{K}_i} \mathbf{w}_i^t (\mathbf{w}_i^t)^T}^2 \\ &= \sum_{i=1}^I \left[\sum_{jt \in \mathbb{I}_t} \|\mathbf{z}_{ijt}^{t+1} - \mathbf{z}_{ijt}^t\|_2^2 - \frac{1}{\tilde{K}_i} \|(\mathbf{w}_i^t)^T (\mathbf{z}_i^{t+1} - \mathbf{z}_i^t)\|_2^2 \right] \\ &= \sum_{i=1}^I \left[\|\mathbf{z}_i^{t+1} - \mathbf{z}_i^t\|_2^2 - \frac{1}{\tilde{K}_i} \|\mathbf{A}_i^T (\mathbf{x}^{t+1} - \mathbf{x}^t)\|_2^2 \right]. \end{aligned} \quad (49)$$

In PDMM, an index set \mathbb{I}_t is randomly chosen. Conditioned on \mathbf{x}^t , \mathbf{x}^{t+1} and \mathbf{y}^{t+1} depend on \mathbb{I}_t . \mathbf{P}_t depends on \mathbb{I}_t . $\mathbf{x}^t, \mathbf{y}^t$ are independent of \mathbb{I}_t . \mathbf{x}^t depends on a sequence of observed realization of random variable

$$\xi_{t-1} = \{\mathbb{I}_1, \mathbb{I}_2, \dots, \mathbb{I}_{t-1}\}. \quad (50)$$

As we do not assume that f_{j_t} is differentiable, we use the subgradient of f_{j_t} . In particular, if f_{j_t} is differentiable, the subgradient of f_{j_t} becomes the gradient, i.e., $\nabla f_{j_t}(\mathbf{x}_{j_t})$. PDMM (5)-(7) has the following lemma.

Lemma 4 Let $\{\mathbf{x}_{j_t}^t, \mathbf{y}_i^t\}$ be generated by PDMM (5)-(7). Assume $\tau_i > 0$ and $\nu_i \geq 0$. We have

$$\begin{aligned}
& \sum_{j_t \in \mathbb{I}_t} f_{j_t}(\mathbf{x}_{j_t}^{t+1}) - f_{j_t}(\mathbf{x}_{j_t}^*) \leq -\frac{K}{J} \sum_{i=1}^I \left\{ \langle \mathbf{y}_i^t, \mathbf{A}_i^r \mathbf{x}^t - \mathbf{a}_i \rangle - \frac{\tau_i \rho}{2} \|\mathbf{A}_i^r \mathbf{x}^t - \mathbf{a}_i\|_2^2 \right\} \\
& - \sum_{j_t \in \mathbb{I}_t} \langle \hat{\mathbf{y}}^t + \rho(\mathbf{A} \mathbf{x}^t - \mathbf{a}), \mathbf{A}_{j_t}^c(\mathbf{x}_{j_t}^t - \mathbf{x}_{j_t}^*) \rangle + \frac{K}{J} \langle \hat{\mathbf{y}}^t + \rho(\mathbf{A} \mathbf{x}^t - \mathbf{a}), \mathbf{A} \mathbf{x}^t - \mathbf{a} \rangle \\
& + \sum_{i=1}^I \left\{ \langle \mathbf{y}_i^t, \mathbf{A}_i^r \mathbf{x}^t - \mathbf{a}_i \rangle - \frac{\tau_i \rho}{2} \|\mathbf{A}_i^r \mathbf{x}^t - \mathbf{a}_i\|_2^2 \right\} - \sum_{i=1}^I \left\{ \langle \mathbf{y}_i^{t+1}, \mathbf{A}_i^r \mathbf{x}^{t+1} - \mathbf{a}_i \rangle - \frac{\tau_i \rho}{2} \|\mathbf{A}_i^r \mathbf{x}^{t+1} - \mathbf{a}_i\|_2^2 \right\} \\
& + \frac{\rho}{2} (\|\mathbf{z}^* - \mathbf{z}^t\|_{\mathbf{Q}}^2 - \|\mathbf{z}^* - \mathbf{z}^{t+1}\|_{\mathbf{Q}}^2 - \|\mathbf{z}^{t+1} - \mathbf{z}^t\|_{\mathbf{P}_t}^2) \\
& + \sum_{j_t \in \mathbb{I}_t} \eta_{j_t} (B_{\phi_{j_t}}(\mathbf{x}_{j_t}^*, \mathbf{x}_{j_t}^t) - B_{\phi_{j_t}}(\mathbf{x}_{j_t}^*, \mathbf{x}_{j_t}^{t+1}) - B_{\phi_{j_t}}(\mathbf{x}_{j_t}^{t+1}, \mathbf{x}_{j_t}^t)) \\
& + \frac{\rho}{2} \sum_{i=1}^I \left\{ \left[(1 - \frac{2K}{J})(1 - \nu_i) + (1 - \frac{K}{J})\tau_i + \frac{1}{d_i} \right] \|\mathbf{A}_i^r \mathbf{x}^t - \mathbf{a}_i\|_2^2 - (1 - \nu_i - \tau_i + \frac{1}{d_i}) \|\mathbf{A}_i^r \mathbf{x}^{t+1} - \mathbf{a}_i\|_2^2 \right. \\
& \left. + (1 - \nu_i - \frac{1}{K_i}) \|\mathbf{A}_i^r(\mathbf{x}^{t+1} - \mathbf{x}^t)\|_2^2 \right\}. \tag{51}
\end{aligned}$$

Proof: Let $\partial f_{j_t}(\mathbf{x}_{j_t}^{t+1})$ be the subdifferential of f_{j_t} at $\mathbf{x}_{j_t}^{t+1}$. The optimality of the \mathbf{x}_{j_t} update (5) is

$$0 \in \partial f_{j_t}(\mathbf{x}_{j_t}^{t+1}) + (\mathbf{A}_{j_t}^c)^T [\hat{\mathbf{y}}^t + \rho(\mathbf{A}_{j_t}^c \mathbf{x}_{j_t}^{t+1} + \sum_{k \neq j_t} \mathbf{A}_k^c \mathbf{x}_k^t - \mathbf{a})] + \eta_{j_t} (\nabla \phi_{j_t}(\mathbf{x}_{j_t}^{t+1}) - \nabla \phi_{j_t}(\mathbf{x}_{j_t}^t)), \tag{52}$$

Using (7) and rearranging the terms yield

$$- (\mathbf{A}_{j_t}^c)^T [\hat{\mathbf{y}}^t + \rho(\mathbf{A} \mathbf{x}^t - \mathbf{a}) + \rho \mathbf{A}_{j_t}^c (\mathbf{x}_{j_t}^{t+1} - \mathbf{x}_{j_t}^t)] + \eta_{j_t} (\nabla \phi_{j_t}(\mathbf{x}_{j_t}^{t+1}) - \nabla \phi_{j_t}(\mathbf{x}_{j_t}^t)) \in \partial f_{j_t}(\mathbf{x}_{j_t}^{t+1}). \tag{53}$$

Using the convexity of f_{j_t} , we have

$$\begin{aligned}
& f_{j_t}(\mathbf{x}_{j_t}^{t+1}) - f_{j_t}(\mathbf{x}_{j_t}^*) \leq -\langle \hat{\mathbf{y}}^t + \rho(\mathbf{A} \mathbf{x}^t - \mathbf{a}), \mathbf{A}_{j_t}^c(\mathbf{x}_{j_t}^{t+1} - \mathbf{x}_{j_t}^*) \rangle \\
& - \rho \langle \mathbf{A}_{j_t}^c(\mathbf{x}_{j_t}^{t+1} - \mathbf{x}_{j_t}^t), \mathbf{A}_{j_t}^c(\mathbf{x}_{j_t}^{t+1} - \mathbf{x}_{j_t}^*) \rangle - \eta_{j_t} \langle \nabla \phi_{j_t}(\mathbf{x}_{j_t}^{t+1}) - \nabla \phi_{j_t}(\mathbf{x}_{j_t}^t), \mathbf{x}_{j_t}^{t+1} - \mathbf{x}_{j_t}^* \rangle \\
& = -\langle \hat{\mathbf{y}}^t + \rho(\mathbf{A} \mathbf{x}^t - \mathbf{a}), \mathbf{A}_{j_t}^c(\mathbf{x}_{j_t}^t - \mathbf{x}_{j_t}^*) \rangle - \langle \hat{\mathbf{y}}^t + \rho(\mathbf{A} \mathbf{x}^t - \mathbf{a}), \mathbf{A}_{j_t}^c(\mathbf{x}_{j_t}^{t+1} - \mathbf{x}_{j_t}^t) \rangle \\
& - \rho \sum_{i=1}^I \langle \mathbf{A}_{ij_t}(\mathbf{x}_{j_t}^{t+1} - \mathbf{x}_{j_t}^t), \mathbf{A}_{ij_t}(\mathbf{x}_{j_t}^{t+1} - \mathbf{x}_{j_t}^*) \rangle \\
& + \eta_{j_t} \left(B_{\phi_{j_t}}(\mathbf{x}_{j_t}^*, \mathbf{x}_{j_t}^t) - B_{\phi_{j_t}}(\mathbf{x}_{j_t}^*, \mathbf{x}_{j_t}^{t+1}) - B_{\phi_{j_t}}(\mathbf{x}_{j_t}^{t+1}, \mathbf{x}_{j_t}^t) \right). \tag{54}
\end{aligned}$$

Summing over $j_t \in \mathbb{I}_t$, we have

$$\begin{aligned}
& \sum_{j_t \in \mathbb{I}_t} f_{j_t}(\mathbf{x}_{j_t}^{t+1}) - f_{j_t}(\mathbf{x}_{j_t}^*) \\
& \leq - \sum_{j_t \in \mathbb{I}_t} \langle \hat{\mathbf{y}}^t + \rho(\mathbf{A}\mathbf{x}^t - \mathbf{a}), \mathbf{A}_{j_t}^c(\mathbf{x}_{j_t}^t - \mathbf{x}_{j_t}^*) \rangle - \langle \hat{\mathbf{y}}^t + \rho(\mathbf{A}\mathbf{x}^t - \mathbf{a}), \sum_{j_t \in \mathbb{I}_t} \mathbf{A}_{j_t}^c(\mathbf{x}_{j_t}^{t+1} - \mathbf{x}_{j_t}^t) \rangle \\
& \quad - \rho \sum_{i=1}^I \sum_{j_t \in \mathbb{I}_t} \langle \mathbf{A}_{ij_t}(\mathbf{x}_{j_t}^{t+1} - \mathbf{x}_{j_t}^t), \mathbf{A}_{ij_t}(\mathbf{x}_{j_t}^{t+1} - \mathbf{x}_{j_t}^*) \rangle \\
& \quad + \sum_{j_t \in \mathbb{I}_t} \eta_{j_t} \left(B_{\phi_{j_t}}(\mathbf{x}_{j_t}^*, \mathbf{x}_{j_t}^t) - B_{\phi_{j_t}}(\mathbf{x}_{j_t}^*, \mathbf{x}_{j_t}^{t+1}) - B_{\phi_{j_t}}(\mathbf{x}_{j_t}^{t+1}, \mathbf{x}_{j_t}^t) \right) \\
& = - \sum_{j_t \in \mathbb{I}_t} \langle \hat{\mathbf{y}}^t + \rho(\mathbf{A}\mathbf{x}^t - \mathbf{a}), \mathbf{A}_{j_t}^c(\mathbf{x}_{j_t}^t - \mathbf{x}_{j_t}^*) \rangle + \frac{K}{J} \langle \hat{\mathbf{y}}^t + \rho(\mathbf{A}\mathbf{x}^t - \mathbf{a}), \mathbf{A}\mathbf{x}^t - \mathbf{a} \rangle \\
& \quad - \underbrace{\frac{K}{J} \langle \hat{\mathbf{y}}^t + \rho(\mathbf{A}\mathbf{x}^t - \mathbf{a}), \mathbf{A}\mathbf{x}^t - \mathbf{a} \rangle - \langle \hat{\mathbf{y}}^t + \rho(\mathbf{A}\mathbf{x}^t - \mathbf{a}), \mathbf{A}(\mathbf{x}^{t+1} - \mathbf{x}^t) \rangle}_{H_1} \\
& \quad + \underbrace{\frac{\rho}{2} \sum_{i=1}^I \sum_{j_t \in \mathbb{I}_t} (\|\mathbf{A}_{ij_t}(\mathbf{x}_{j_t}^* - \mathbf{x}_{j_t}^t)\|_2^2 - \|\mathbf{A}_{ij_t}(\mathbf{x}_{j_t}^* - \mathbf{x}_{j_t}^{t+1})\|_2^2 - \|\mathbf{A}_{ij_t}(\mathbf{x}_{j_t}^{t+1} - \mathbf{x}_{j_t}^t)\|_2^2)}_{H_2} \\
& \quad + \sum_{j_t \in \mathbb{I}_t} \eta_{j_t} \left(B_{\phi_{j_t}}(\mathbf{x}_{j_t}^*, \mathbf{x}_{j_t}^t) - B_{\phi_{j_t}}(\mathbf{x}_{j_t}^*, \mathbf{x}_{j_t}^{t+1}) - B_{\phi_{j_t}}(\mathbf{x}_{j_t}^{t+1}, \mathbf{x}_{j_t}^t) \right). \tag{55}
\end{aligned}$$

H_1 in (55) can be rewritten as

$$H_1 = -\langle \hat{\mathbf{y}}^t + \rho(\mathbf{A}\mathbf{x}^t - \mathbf{a}), \mathbf{A}\mathbf{x}^{t+1} - \mathbf{a} \rangle + \left(1 - \frac{K}{J}\right) \langle \hat{\mathbf{y}}^t + \rho(\mathbf{A}\mathbf{x}^t - \mathbf{a}), \mathbf{A}\mathbf{x}^t - \mathbf{a} \rangle. \tag{56}$$

The first term of (56) is equivalent to

$$\begin{aligned}
& - \langle \hat{\mathbf{y}}^t + \rho(\mathbf{A}\mathbf{x}^t - \mathbf{a}), \mathbf{A}\mathbf{x}^{t+1} - \mathbf{a} \rangle \\
&= - \sum_{i=1}^I \langle \hat{\mathbf{y}}_i^t + \rho(\mathbf{A}_i^r \mathbf{x}^t - \mathbf{a}_i), \mathbf{A}_i^r \mathbf{x}^{t+1} - \mathbf{a}_i \rangle \\
&= - \sum_{i=1}^I \langle \mathbf{y}_i^t + (1 - \nu_i)\rho(\mathbf{A}_i^r \mathbf{x}^t - \mathbf{a}_i), \mathbf{A}_i^r \mathbf{x}^{t+1} - \mathbf{a}_i \rangle \\
&= - \sum_{i=1}^I \left\{ \langle \mathbf{y}_i^{t+1} - \tau_i \rho(\mathbf{A}_i^r \mathbf{x}^{t+1} - \mathbf{a}_i), \mathbf{A}_i^r \mathbf{x}^{t+1} - \mathbf{a}_i \rangle + (1 - \nu_i)\rho \langle \mathbf{A}_i^r \mathbf{x}^t - \mathbf{a}_i, \mathbf{A}_i^r \mathbf{x}^{t+1} - \mathbf{a}_i \rangle \right\} \\
&= - \sum_{i=1}^I \left\{ \langle \mathbf{y}_i^{t+1}, \mathbf{A}_i^r \mathbf{x}^{t+1} - \mathbf{a}_i \rangle - \tau_i \rho \|\mathbf{A}_i^r \mathbf{x}^{t+1} - \mathbf{a}_i\|_2^2 \right. \\
&\quad \left. - \frac{(1 - \nu_i)\rho}{2} (\|\mathbf{A}_i^r (\mathbf{x}^{t+1} - \mathbf{x}^t)\|_2^2 - \|\mathbf{A}_i^r \mathbf{x}^t - \mathbf{a}_i\|_2^2 - \|\mathbf{A}_i^r \mathbf{x}^{t+1} - \mathbf{a}_i\|_2^2) \right\} \\
&= - \sum_{i=1}^I \left\{ \langle \mathbf{y}_i^{t+1}, \mathbf{A}_i^r \mathbf{x}^{t+1} - \mathbf{a}_i \rangle - \frac{\tau_i \rho}{2} \|\mathbf{A}_i^r \mathbf{x}^{t+1} - \mathbf{a}_i\|_2^2 \right\} \\
&\quad + \sum_{i=1}^I \left\{ \frac{(1 - \nu_i)\rho}{2} (\|\mathbf{A}_i^r (\mathbf{x}^{t+1} - \mathbf{x}^t)\|_2^2 - \|\mathbf{A}_i^r \mathbf{x}^t - \mathbf{a}_i\|_2^2) - \frac{(1 - \nu_i - \tau_i)\rho}{2} \|\mathbf{A}_i^r \mathbf{x}^{t+1} - \mathbf{a}_i\|_2^2 \right\}. \quad (57)
\end{aligned}$$

The second term of (56) is equivalent to

$$\begin{aligned}
& \left(1 - \frac{K}{J}\right) \langle \hat{\mathbf{y}}^t + \rho(\mathbf{A}\mathbf{x}^t - \mathbf{a}), \mathbf{A}\mathbf{x}^t - \mathbf{a} \rangle \\
&= \left(1 - \frac{K}{J}\right) \sum_{i=1}^I \langle \hat{\mathbf{y}}_i^t + \rho(\mathbf{A}_i^r \mathbf{x}^t - \mathbf{a}_i), \mathbf{A}_i^r \mathbf{x}^t - \mathbf{a}_i \rangle \\
&= \left(1 - \frac{K}{J}\right) \sum_{i=1}^I \langle \mathbf{y}_i^t + (1 - \nu_i)\rho(\mathbf{A}_i^r \mathbf{x}^t - \mathbf{a}_i), \mathbf{A}_i^r \mathbf{x}^t - \mathbf{a}_i \rangle \\
&= \left(1 - \frac{K}{J}\right) \sum_{i=1}^I \left\{ \langle \mathbf{y}_i^t, \mathbf{A}_i^r \mathbf{x}^t - \mathbf{a}_i \rangle - \frac{\tau_i \rho}{2} \|\mathbf{A}_i^r \mathbf{x}^t - \mathbf{a}_i\|_2^2 \right\} + \left(1 - \frac{K}{J}\right) \sum_{i=1}^I (1 - \nu_i + \frac{\tau_i}{2}) \rho \|\mathbf{A}_i^r \mathbf{x}^t - \mathbf{a}_i\|_2^2. \quad (58)
\end{aligned}$$

H_2 in (55) is equivalent to

$$\begin{aligned}
H_2 &= \frac{\rho}{2} \sum_{i=1}^I \sum_{j_t \in \mathbb{I}_t} (\|\mathbf{z}_{ij_t}^* - \mathbf{z}_{ij_t}^t\|_2^2 - \|\mathbf{z}_{ij_t}^* - \mathbf{z}_{ij_t}^{t+1}\|_2^2 - \|\mathbf{z}_{ij_t}^{t+1} - \mathbf{z}_{ij_t}^t\|_2^2) \\
&= \frac{\rho}{2} \sum_{i=1}^I (\|\mathbf{z}_i^* - \mathbf{z}_i^t\|_2^2 - \|\mathbf{z}_i^* - \mathbf{z}_i^{t+1}\|_2^2 - \|\mathbf{z}_i^{t+1} - \mathbf{z}_i^t\|_2^2) \\
&= \frac{\rho}{2} (\|\mathbf{z}^* - \mathbf{z}^t\|_{\mathbf{Q}}^2 - \|\mathbf{z}^* - \mathbf{z}^{t+1}\|_{\mathbf{Q}}^2 - \|\mathbf{z}^{t+1} - \mathbf{z}^t\|_{\mathbf{P}_t}^2) \\
&+ \frac{\rho}{2} \sum_{i=1}^I \frac{1}{d_i} (\|\mathbf{A}_i^r \mathbf{x}^t - \mathbf{a}_i\|_2^2 - \|\mathbf{A}_i^r \mathbf{x}^{t+1} - \mathbf{a}_i\|_2^2) - \frac{1}{\tilde{K}_i} \|\mathbf{A}_i^r (\mathbf{x}^{t+1} - \mathbf{x}^t)\|_2^2. \tag{59}
\end{aligned}$$

where the last equality uses the definition of \mathbf{Q} in (42) and \mathbf{P}_t (48), and $\tilde{K}_i = \min\{K, d_i\}$. Combining the results of (56)-(59) gives

$$\begin{aligned}
H_1 + H_2 &= - \sum_{i=1}^I \left\{ \langle \mathbf{y}_i^{t+1}, \mathbf{A}_i^r \mathbf{x}^{t+1} - \mathbf{a}_i \rangle - \frac{\tau_i \rho}{2} \|\mathbf{A}_i^r \mathbf{x}^{t+1} - \mathbf{a}_i\|_2^2 \right\} \\
&+ \sum_{i=1}^I \left\{ \frac{(1 - \nu_i) \rho}{2} (\|\mathbf{A}_i^r (\mathbf{x}^{t+1} - \mathbf{x}^t)\|_2^2 - \|\mathbf{A}_i^r \mathbf{x}^t - \mathbf{a}_i\|_2^2) - \frac{(1 - \nu_i - \tau_i) \rho}{2} \|\mathbf{A}_i^r \mathbf{x}^{t+1} - \mathbf{a}_i\|_2^2 \right\} \\
&+ (1 - \frac{K}{J}) \sum_{i=1}^I \left\{ \langle \mathbf{y}_i^t, \mathbf{A}_i^r \mathbf{x}^t - \mathbf{a}_i \rangle - \frac{\tau_i \rho}{2} \|\mathbf{A}_i^r \mathbf{x}^t - \mathbf{a}_i\|_2^2 \right\} + (1 - \frac{K}{J}) \sum_{i=1}^I (1 - \nu_i + \frac{\tau_i}{2}) \rho \|\mathbf{A}_i^r \mathbf{x}^t - \mathbf{a}_i\|_2^2 \\
&+ \frac{\rho}{2} (\|\mathbf{z}^* - \mathbf{z}^t\|_{\mathbf{Q}}^2 - \|\mathbf{z}^* - \mathbf{z}^{t+1}\|_{\mathbf{Q}}^2 - \|\mathbf{z}^{t+1} - \mathbf{z}^t\|_{\mathbf{P}_t}^2) \\
&+ \frac{\rho}{2} \sum_{i=1}^I \frac{1}{d_i} (\|\mathbf{A}_i^r \mathbf{x}^t - \mathbf{a}_i\|_2^2 - \|\mathbf{A}_i^r \mathbf{x}^{t+1} - \mathbf{a}_i\|_2^2) - \frac{1}{\tilde{K}_i} \|\mathbf{A}_i^r (\mathbf{x}^{t+1} - \mathbf{x}^t)\|_2^2 \\
&= - \frac{K}{J} \sum_{i=1}^I \left\{ \langle \mathbf{y}_i^t, \mathbf{A}_i^r \mathbf{x}^t - \mathbf{a}_i \rangle - \frac{\tau_i \rho}{2} \|\mathbf{A}_i^r \mathbf{x}^t - \mathbf{a}_i\|_2^2 \right\} \\
&+ \sum_{i=1}^I \left\{ \langle \mathbf{y}_i^t, \mathbf{A}_i^r \mathbf{x}^t - \mathbf{a}_i \rangle - \frac{\tau_i \rho}{2} \|\mathbf{A}_i^r \mathbf{x}^t - \mathbf{a}_i\|_2^2 \right\} - \sum_{i=1}^I \left\{ \langle \mathbf{y}_i^{t+1}, \mathbf{A}_i^r \mathbf{x}^{t+1} - \mathbf{a}_i \rangle - \frac{\tau_i \rho}{2} \|\mathbf{A}_i^r \mathbf{x}^{t+1} - \mathbf{a}_i\|_2^2 \right\} \\
&+ \frac{\rho}{2} (\|\mathbf{z}^* - \mathbf{z}^t\|_{\mathbf{Q}}^2 - \|\mathbf{z}^* - \mathbf{z}^{t+1}\|_{\mathbf{Q}}^2 - \|\mathbf{z}^{t+1} - \mathbf{z}^t\|_{\mathbf{P}_t}^2) \\
&+ \frac{\rho}{2} \sum_{i=1}^I \left\{ [(1 - \frac{2K}{J})(1 - \nu_i) + (1 - \frac{K}{J})\tau_i + \frac{1}{d_i}] \|\mathbf{A}_i^r \mathbf{x}^t - \mathbf{a}_i\|_2^2 - (1 - \nu_i - \tau_i + \frac{1}{d_i}) \|\mathbf{A}_i^r \mathbf{x}^{t+1} - \mathbf{a}_i\|_2^2 \right. \\
&\left. + (1 - \nu_i - \frac{1}{\tilde{K}_i}) \|\mathbf{A}_i^r (\mathbf{x}^{t+1} - \mathbf{x}^t)\|_2^2 \right\}. \tag{60}
\end{aligned}$$

Plugging back into (55) completes the proof. \blacksquare

Lemma 5 Let $\{\mathbf{x}_{j_t}^t, \mathbf{y}_i^t\}$ be generated by PDMM (5)-(7). Assume $\tau_i > 0$ and $\nu_i \geq 0$. We have

$$\begin{aligned}
& \sum_{j_t \in \mathbb{I}_t} f_{j_t}(\mathbf{x}_{j_t}^{t+1}) - f_{j_t}(\mathbf{x}_{j_t}^*) \leq -\frac{K}{J} \sum_{i=1}^I \left\{ \langle \mathbf{y}_i^t, \mathbf{A}_i^r \mathbf{x}^t - \mathbf{a}_i \rangle - \frac{\tau_i \rho}{2} \|\mathbf{A}_i^r \mathbf{x}^t - \mathbf{a}_i\|_2^2 \right\} \\
& - \sum_{j_t \in \mathbb{I}_t} \langle \hat{\mathbf{y}}^t + \rho(\mathbf{A} \mathbf{x}^t - \mathbf{a}), \mathbf{A}_{j_t}^c (\mathbf{x}_{j_t}^t - \mathbf{x}_{j_t}^*) \rangle + \frac{K}{J} \langle \hat{\mathbf{y}}^t + \rho(\mathbf{A} \mathbf{x}^t - \mathbf{a}), \mathbf{A} \mathbf{x}^t - \mathbf{a} \rangle \\
& + \sum_{i=1}^I \left\{ \langle \mathbf{y}_i^t, \mathbf{A}_i^r \mathbf{x}^t - \mathbf{a}_i \rangle - \frac{\tau_i \rho}{2} \|\mathbf{A}_i^r \mathbf{x}^t - \mathbf{a}_i\|_2^2 \right\} - \sum_{i=1}^I \left\{ \langle \mathbf{y}_i^{t+1}, \mathbf{A}_i^r \mathbf{x}^{t+1} - \mathbf{a}_i \rangle - \frac{\tau_i \rho}{2} \|\mathbf{A}_i^r \mathbf{x}^{t+1} - \mathbf{a}_i\|_2^2 \right\} \\
& + \frac{\rho}{2} (\|\mathbf{z}^* - \mathbf{z}^t\|_{\mathbf{Q}}^2 - \|\mathbf{z}^* - \mathbf{z}^{t+1}\|_{\mathbf{Q}}^2 - \|\mathbf{z}^{t+1} - \mathbf{z}^t\|_{\mathbf{P}_t}^2) \\
& + \boldsymbol{\eta}^T (B_\phi(\mathbf{x}^*, \mathbf{x}^t) - B_\phi(\mathbf{x}^*, \mathbf{x}^{t+1}) - B_\phi(\mathbf{x}^{t+1}, \mathbf{x}^t)) \\
& + \frac{\rho}{2} \sum_{i=1}^I [\gamma_i (\|\mathbf{A}_i^r \mathbf{x}^t - \mathbf{a}_i\|_2^2 - \|\mathbf{A}_i^r \mathbf{x}^{t+1} - \mathbf{a}_i\|_2^2) - \beta_i \|\mathbf{A}_i^r \mathbf{x}^{t+1} - \mathbf{a}_i\|_2^2] . \tag{61}
\end{aligned}$$

where $\boldsymbol{\eta}^T = [\eta_1, \dots, \eta_J]$. $\tau_i > 0, \nu_i \geq 0, \gamma_i \geq 0$ and $\beta_i \geq 0$ satisfy the following conditions:

$$\nu_i \in \left(\max\left\{0, 1 - \frac{2J}{\tilde{K}_i(2J - K)}\right\}, 1 - \frac{1}{\tilde{K}_i} \right], \tag{62}$$

$$\tau_i \leq \frac{J}{2J - K} \left[\frac{4}{\tilde{K}_i} - \left(4 - \frac{2K}{J}\right) (1 - \nu_i) \right] \leq \frac{2K}{\tilde{K}_i(2J - K)}, \tag{63}$$

$$\gamma_i = \left(3 - \frac{2K}{J}\right) (1 - \nu_i) + \left(1 - \frac{K}{J}\right) \tau_i + \frac{1}{d_i} - \frac{2}{\tilde{K}_i}, \tag{64}$$

$$\beta_i = \frac{4}{\tilde{K}_i} - \left(2 - \frac{K}{J}\right) [2(1 - \nu_i) + \tau_i]. \tag{65}$$

Proof: In (51), denote

$$H_3 = \left[\left(1 - \frac{2K}{J}\right) (1 - \nu_i) + \left(1 - \frac{K}{J}\right) \tau_i + \frac{1}{d_i} \right] \|\mathbf{A}_i^r \mathbf{x}^t - \mathbf{a}_i\|_2^2 - \left(1 - \nu_i - \tau_i + \frac{1}{d_i}\right) \|\mathbf{A}_i^r \mathbf{x}^{t+1} - \mathbf{a}_i\|_2^2, \tag{66}$$

$$H_4 = \left(1 - \nu_i - \frac{1}{\tilde{K}_i}\right) \|\mathbf{A}_i^r (\mathbf{x}^{t+1} - \mathbf{x}^t)\|_2^2. \tag{67}$$

Our goal is to eliminate H_4 so that

$$H_3 + H_4 = \gamma_i (\|\mathbf{A}_i^r \mathbf{x}^t - \mathbf{a}_i\|_2^2 - \|\mathbf{A}_i^r \mathbf{x}^{t+1} - \mathbf{a}_i\|_2^2) - \beta_i \|\mathbf{A}_i^r \mathbf{x}^{t+1} - \mathbf{a}_i\|_2^2, \tag{68}$$

where $\gamma_i \geq 0$ and $\beta_i \geq 0$.

We want to choose a large τ_i and a small ν_i . Assume $1 - \nu_i - \frac{1}{\tilde{K}_i} \geq 0$, i.e., $\nu_i \leq 1 - \frac{1}{\tilde{K}_i}$, we have

$$H_4 = \left(1 - \nu_i - \frac{1}{\tilde{K}_i}\right) \|\mathbf{A}_i^r (\mathbf{x}^{t+1} - \mathbf{x}^t)\|_2^2 \leq 2 \left(1 - \nu_i - \frac{1}{\tilde{K}_i}\right) (\|\mathbf{A}_i^r \mathbf{x}^t - \mathbf{a}_i\|_2^2 + \|\mathbf{A}_i^r \mathbf{x}^{t+1} - \mathbf{a}_i\|_2^2). \tag{69}$$

Therefore, we have

$$\begin{aligned}
H_3 + H_4 & \leq \left[\left(3 - \frac{2K}{J}\right) (1 - \nu_i) + \left(1 - \frac{K}{J}\right) \tau_i + \frac{1}{d_i} - \frac{2}{\tilde{K}_i} \right] \|\mathbf{A}_i^r \mathbf{x}^t - \mathbf{a}_i\|_2^2 + \left(1 - \nu_i + \tau_i - \frac{1}{d_i} - \frac{2}{\tilde{K}_i}\right) \|\mathbf{A}_i^r \mathbf{x}^{t+1} - \mathbf{a}_i\|_2^2 \\
& = \gamma_i (\|\mathbf{A}_i^r \mathbf{x}^t - \mathbf{a}_i\|_2^2 - \|\mathbf{A}_i^r \mathbf{x}^{t+1} - \mathbf{a}_i\|_2^2) - \beta_i \|\mathbf{A}_i^r \mathbf{x}^{t+1} - \mathbf{a}_i\|_2^2. \tag{70}
\end{aligned}$$

where

$$\begin{aligned}
\gamma_i &= \left(3 - \frac{2K}{J}\right)(1 - \nu_i) + \left(1 - \frac{K}{J}\right)\tau_i + \frac{1}{d_i} - \frac{2}{\tilde{K}_i} \\
&\geq \left(3 - \frac{2K}{J}\right)\frac{1}{\tilde{K}_i} + \left(1 - \frac{K}{J}\right)\tau_i + \frac{1}{d_i} - \frac{2}{\tilde{K}_i} \\
&= \left(1 - \frac{K}{J}\right)\frac{1}{\tilde{K}_i} - \frac{K}{J\tilde{K}_i} + \frac{1}{d_i} + \left(1 - \frac{K}{J}\right)\tau_i \geq 0.
\end{aligned} \tag{71}$$

and

$$\beta_i = -(1 - \nu_i + \tau_i - \frac{1}{d_i} - \frac{2}{\tilde{K}_i} + \gamma_i) = \frac{4}{\tilde{K}_i} - \left(2 - \frac{K}{J}\right)[2(1 - \nu_i) + \tau_i]. \tag{72}$$

We also want $\beta_i \geq 0$, which can be reduced to

$$\begin{aligned}
\tau_i &\leq \frac{J}{2J - K} \left[\frac{4}{\tilde{K}_i} - \left(4 - \frac{2K}{J}\right)(1 - \nu_i) \right] \\
&\leq \frac{J}{2J - K} \left[\frac{4}{\tilde{K}_i} - \left(4 - \frac{2K}{J}\right)\frac{1}{\tilde{K}_i} \right] \\
&= \frac{2K}{\tilde{K}_i(2J - K)}.
\end{aligned} \tag{73}$$

It also requires the RHS of (73) to be positive, leading to $\nu_i > \max\{0, 1 - \frac{2J}{\tilde{K}_i(2J - K)}\}$. Therefore, $\nu_i \in (\max\{0, 1 - \frac{2J}{\tilde{K}_i(2J - K)}\}, 1 - \frac{1}{\tilde{K}_i}]$.

Denote $B_\phi = [B_{\phi_1}, \dots, B_{\phi_J}]^T$ as a column vector of the Bregman divergence on block coordinates of \mathbf{x} . Using $\mathbf{x}^{t+1} = [\mathbf{x}_{j_t \in \mathbb{I}_t}^{t+1}, \mathbf{x}_{j_t \notin \mathbb{I}_t}^t]^T$, we have $B_{\phi_{j_t}}(\mathbf{x}_{j_t}^*, \mathbf{x}_{j_t}^t) - B_{\phi_{j_t}}(\mathbf{x}_{j_t}^*, \mathbf{x}_{j_t}^{t+1}) = B_\phi(\mathbf{x}^*, \mathbf{x}^t) - B_\phi(\mathbf{x}^*, \mathbf{x}^{t+1})$, $B_{\phi_{j_t}}(\mathbf{x}_{j_t}^{t+1}, \mathbf{x}_{j_t}^t) = B_\phi(\mathbf{x}^{t+1}, \mathbf{x}^t)$. Thus,

$$\begin{aligned}
&\sum_{j_t \in \mathbb{I}_t} \eta_{j_t} \left(B_{\phi_{j_t}}(\mathbf{x}_{j_t}^*, \mathbf{x}_{j_t}^t) - B_{\phi_{j_t}}(\mathbf{x}_{j_t}^*, \mathbf{x}_{j_t}^{t+1}) - B_{\phi_{j_t}}(\mathbf{x}_{j_t}^{t+1}, \mathbf{x}_{j_t}^t) \right) \\
&= \boldsymbol{\eta}^T (B_\phi(\mathbf{x}^*, \mathbf{x}^t) - B_\phi(\mathbf{x}^*, \mathbf{x}^{t+1}) - B_\phi(\mathbf{x}^{t+1}, \mathbf{x}^t)).
\end{aligned} \tag{74}$$

where $\boldsymbol{\eta}^T = [\eta_1, \dots, \eta_J]$. ■

Lemma 6 Let $\{\mathbf{x}_{j_t}^t, \mathbf{y}_i^t\}$ be generated by PDMM (5)-(7). Assume $\tau_i > 0$ and $\nu_i \geq 0$ satisfy the conditions in Lemma 5. We have

$$\begin{aligned}
f(\mathbf{x}^t) - f(\mathbf{x}^*) &\leq - \sum_{i=1}^I \left\{ \langle \mathbf{y}_i^t, \mathbf{A}_i^r \mathbf{x}^t - \mathbf{a}_i \rangle - \frac{\tau_i \rho}{2} \|\mathbf{A}_i^r \mathbf{x}^t - \mathbf{a}_i\|_2^2 \right\} \\
&+ \frac{J}{K} \left\{ \tilde{\mathcal{L}}_\rho(\mathbf{x}^t, \mathbf{y}^t) - \mathbb{E}_{\mathbb{I}_t} \tilde{\mathcal{L}}_\rho(\mathbf{x}^{t+1}, \mathbf{y}^{t+1}) - \frac{\rho}{2} \sum_{i=1}^I \beta_i \mathbb{E}_{\mathbb{I}_t} \|\mathbf{A}_i^r \mathbf{x}^{t+1} - \mathbf{a}_i\|_2^2 \right. \\
&+ \frac{\rho}{2} (\|\mathbf{z}^* - \mathbf{z}^t\|_{\mathbf{Q}}^2 - \mathbb{E}_{\mathbb{I}_t} \|\mathbf{z}^* - \mathbf{z}^{t+1}\|_{\mathbf{Q}}^2 - \mathbb{E}_{\mathbb{I}_t} \|\mathbf{z}^{t+1} - \mathbf{z}^t\|_{\mathbf{P}_t}^2) \\
&\left. + \boldsymbol{\eta}^T (B_\phi(\mathbf{x}^*, \mathbf{x}^t) - \mathbb{E}_{\mathbb{I}_t} B_\phi(\mathbf{x}^*, \mathbf{x}^{t+1}) - \mathbb{E}_{\mathbb{I}_t} B_\phi(\mathbf{x}^{t+1}, \mathbf{x}^t)) \right\}.
\end{aligned} \tag{75}$$

where $\tilde{\mathcal{L}}_\rho$ is defined as follows:

$$\tilde{\mathcal{L}}_\rho(\mathbf{x}^t, \mathbf{y}^t) = f(\mathbf{x}^t) - f(\mathbf{x}^*) + \sum_{i=1}^I \left\{ \langle \mathbf{y}_i^t, \mathbf{A}_i^r \mathbf{x}^t - \mathbf{a}_i \rangle + \frac{(\gamma_i - \tau_i)\rho}{2} \|\mathbf{A}_i^r \mathbf{x}^t - \mathbf{a}_i\|_2^2 \right\}. \quad (76)$$

$\tau_i, \nu_i, \gamma_i, \beta_i$ and $\boldsymbol{\eta}$ are defined in Lemma 5.

Proof: Using $\mathbf{x}^{t+1} = [\mathbf{x}_{j_t \in \mathbb{I}_t}^{t+1}, \mathbf{x}_{j_t \notin \mathbb{I}_t}^t]^T$, we have

$$f(\mathbf{x}^{t+1}) - f(\mathbf{x}^t) = \sum_{j_t \in \mathbb{I}_t} f_{j_t}(\mathbf{x}_{j_t}^{t+1}) - f_{j_t}(\mathbf{x}_{j_t}^t) = \sum_{j_t \in \mathbb{I}_t} [f_{j_t}(\mathbf{x}_{j_t}^{t+1}) - f_{j_t}(\mathbf{x}_{j_t}^*)] - \sum_{j_t \in \mathbb{I}_t} [f_{j_t}(\mathbf{x}_{j_t}^t) - f_{j_t}(\mathbf{x}_{j_t}^*)]. \quad (77)$$

Rearranging the terms and using Lemma 5 yield

$$\begin{aligned} & \sum_{j_t \in \mathbb{I}_t} f_{j_t}(\mathbf{x}_{j_t}^t) - f_{j_t}(\mathbf{x}_{j_t}^*) = \sum_{j_t \in \mathbb{I}_t} [f_{j_t}(\mathbf{x}_{j_t}^{t+1}) - f_{j_t}(\mathbf{x}_{j_t}^*)] + f(\mathbf{x}^t) - f(\mathbf{x}^{t+1}) \\ & \leq -\frac{K}{J} \sum_{i=1}^I \left\{ \langle \mathbf{y}_i^t, \mathbf{A}_i^r \mathbf{x}^t - \mathbf{a}_i \rangle - \frac{\tau_i \rho}{2} \|\mathbf{A}_i^r \mathbf{x}^t - \mathbf{a}_i\|_2^2 \right\} \\ & \quad - \sum_{j_t \in \mathbb{I}_t} \langle \hat{\mathbf{y}}^t + \rho(\mathbf{A}\mathbf{x}^t - \mathbf{a}), \mathbf{A}_{j_t}^c(\mathbf{x}_{j_t}^t - \mathbf{x}_{j_t}^*) \rangle + \frac{K}{J} \langle \hat{\mathbf{y}}^t + \rho(\mathbf{A}\mathbf{x}^t - \mathbf{a}), \mathbf{A}\mathbf{x}^t - \mathbf{a} \rangle \\ & \quad + \tilde{\mathcal{L}}_\rho(\mathbf{x}^t, \mathbf{y}^t) - \tilde{\mathcal{L}}_\rho(\mathbf{x}^{t+1}, \mathbf{y}^{t+1}) - \frac{\rho}{2} \sum_{i=1}^I \beta_i \|\mathbf{A}_i^r \mathbf{x}^{t+1} - \mathbf{a}_i\|_2^2 \\ & \quad + \frac{\rho}{2} (\|\mathbf{z}^* - \mathbf{z}^t\|_{\mathbf{Q}}^2 - \|\mathbf{z}^* - \mathbf{z}^{t+1}\|_{\mathbf{Q}}^2 - \|\mathbf{z}^{t+1} - \mathbf{z}^t\|_{\mathbf{P}_t}^2) \\ & \quad + \boldsymbol{\eta}^T (B_\phi(\mathbf{x}^*, \mathbf{x}^t) - B_\phi(\mathbf{x}^*, \mathbf{x}^{t+1}) - B_\phi(\mathbf{x}^{t+1}, \mathbf{x}^t)), \end{aligned} \quad (78)$$

where $\tilde{\mathcal{L}}_\rho(\mathbf{x}^t, \mathbf{y}^t)$ is defined in (76). Conditioning on \mathbf{x}^t and taking expectation over \mathbb{I}_t , we have

$$\begin{aligned} & \frac{K}{J} [f(\mathbf{x}^t) - f(\mathbf{x}^*)] \leq -\frac{K}{J} \sum_{i=1}^I \left\{ \langle \mathbf{y}_i^t, \mathbf{A}_i^r \mathbf{x}^t - \mathbf{a}_i \rangle - \frac{\tau_i \rho}{2} \|\mathbf{A}_i^r \mathbf{x}^t - \mathbf{a}_i\|_2^2 \right\} \\ & \quad + \tilde{\mathcal{L}}_\rho(\mathbf{x}^t, \mathbf{y}^t) - \mathbb{E}_{\mathbb{I}_t} \tilde{\mathcal{L}}_\rho(\mathbf{x}^{t+1}, \mathbf{y}^{t+1}) - \frac{\rho}{2} \sum_{i=1}^I \beta_i \mathbb{E}_{\mathbb{I}_t} \|\mathbf{A}_i^r \mathbf{x}^{t+1} - \mathbf{a}_i\|_2^2 \\ & \quad + \frac{\rho}{2} (\|\mathbf{z}^* - \mathbf{z}^t\|_{\mathbf{Q}}^2 - \mathbb{E}_{\mathbb{I}_t} \|\mathbf{z}^* - \mathbf{z}^{t+1}\|_{\mathbf{Q}}^2 - \mathbb{E}_{\mathbb{I}_t} \|\mathbf{z}^{t+1} - \mathbf{z}^t\|_{\mathbf{P}_t}^2) \\ & \quad + \boldsymbol{\eta}^T (B_\phi(\mathbf{x}^*, \mathbf{x}^t) - \mathbb{E}_{\mathbb{I}_t} B_\phi(\mathbf{x}^*, \mathbf{x}^{t+1}) - \mathbb{E}_{\mathbb{I}_t} B_\phi(\mathbf{x}^{t+1}, \mathbf{x}^t)), \end{aligned} \quad (79)$$

where we use

$$\mathbb{E}_{\mathbb{I}_t} \left[- \sum_{j_t \in \mathbb{I}_t} \langle \hat{\mathbf{y}}^t + \rho(\mathbf{A}\mathbf{x}^t - \mathbf{a}), \mathbf{A}_{j_t}^c(\mathbf{x}_{j_t}^t - \mathbf{x}_{j_t}^*) \rangle \right] = -\frac{K}{J} \langle \hat{\mathbf{y}}^t + \rho(\mathbf{A}\mathbf{x}^t - \mathbf{a}), \mathbf{A}\mathbf{x}^t - \mathbf{a} \rangle. \quad (80)$$

Dividing both sides by $\frac{K}{J}$ and using the definition (76) complete the proof. \blacksquare

A.2 Theoretical Results

We establish the convergence results for PDMM under fairly simple assumptions:

Assumption 2

- (1) $f_j : \mathbb{R}^{n_j} \rightarrow \mathbb{R} \cup \{+\infty\}$ are closed, proper, and convex.
- (2) A KKT point of the Lagrangian ($\rho = 0$ in (2)) of Problem (1) exists.

Assumption 2 is the same as that required by ADMM [3, 35]. Let ∂f_j be the subdifferential of f_j . Assume that $\{\mathbf{x}_j^*, \mathbf{y}_i^*\}$ satisfies the KKT conditions of the Lagrangian ($\rho = 0$ in (2)), i.e.,

$$-\mathbf{A}_j^T \mathbf{y}^* \in \partial f_j(\mathbf{x}_j^*), \quad (81)$$

$$\mathbf{A}\mathbf{x}^* - \mathbf{a} = 0. \quad (82)$$

During iterations, (82) is satisfied if $\mathbf{A}\mathbf{x}^{t+1} = \mathbf{a}$. The optimality conditions for the \mathbf{x}_j update (5) is

$$0 \in \partial f_j(\mathbf{x}_j^{t+1}) + \mathbf{A}_j^c[\hat{\mathbf{y}}^t + \rho(\mathbf{A}_j^c \mathbf{x}_j^{t+1} + \sum_{k \neq j} \mathbf{A}_k^c \mathbf{x}_k^t - \mathbf{a})] + \eta_j(\nabla \phi_j(\mathbf{x}_j^{t+1}) - \nabla \phi_j(\mathbf{x}_j^t)), \quad (83)$$

which is equivalent to

$$-\mathbf{A}_j^c[\mathbf{y}^t + (1 - \nu)\rho(\mathbf{A}\mathbf{x}^t - \mathbf{a}) + \mathbf{A}_j^c(\mathbf{x}_j^{t+1} - \mathbf{x}_j^t)] - \eta_j(\nabla \phi_j(\mathbf{x}_j^{t+1}) - \nabla \phi_j(\mathbf{x}_j^t)) \in \partial f_j(\mathbf{x}_j^{t+1}). \quad (84)$$

When $\mathbf{A}\mathbf{x}^{t+1} = \mathbf{a}$, $\mathbf{y}^{t+1} = \mathbf{y}^t$. If $\mathbf{A}_j^c(\mathbf{x}_j^{t+1} - \mathbf{x}_j^t) = 0$, then $\mathbf{A}\mathbf{x}^t - \mathbf{a} = 0$. When $\eta_j \geq 0$, further assuming $B_{\phi_j}(\mathbf{x}_j^{t+1}, \mathbf{x}_j^t) = 0$, (81) will be satisfied. Overall, the KKT conditions (81)-(82) are satisfied if the following optimality conditions are satisfied by the iterates:

$$\mathbf{A}\mathbf{x}^{t+1} = \mathbf{a}, \mathbf{A}_j^c(\mathbf{x}_j^{t+1} - \mathbf{x}_j^t) = 0, \quad (85)$$

$$B_{\phi_j}(\mathbf{x}_j^{t+1}, \mathbf{x}_j^t) = 0. \quad (86)$$

The above optimality conditions are sufficient for the KKT conditions. (85) are the optimality conditions for the exact PDMM. (86) is needed only when $\eta_j > 0$.

In Lemma 5, setting the values of $\nu_i, \tau_i, \gamma_i, \beta_i$ as follows:

$$\nu_i = 1 - \frac{1}{\tilde{K}_i}, \tau_i = \frac{K}{\tilde{K}_i(2J - K)}, \gamma_i = \frac{2(J - K)}{\tilde{K}_i(2J - K)} + \frac{1}{d_i} - \frac{K}{J\tilde{K}_i}, \beta_i = \frac{K}{J\tilde{K}_i}. \quad (87)$$

Define the residual of optimality conditions (85)-(86) as

$$R(\mathbf{x}^{t+1}) = \frac{\rho}{2} \|\mathbf{z}^{t+1} - \mathbf{z}^t\|_{\mathbf{P}_t}^2 + \frac{\rho}{2} \sum_{i=1}^I \beta_i \|\mathbf{A}_i^T \mathbf{x}^{t+1} - \mathbf{a}_i\|_2^2 + [\boldsymbol{\eta}^T B_\phi(\mathbf{x}^{t+1}, \mathbf{x}^t)]. \quad (88)$$

If $R(\mathbf{x}^{t+1}) \rightarrow 0$, (85)-(86) will be satisfied and thus PDMM converges to the KKT point $\{\mathbf{x}^*, \mathbf{y}^*\}$.

Define the current iterate $\mathbf{v}^t = (\mathbf{x}_j^t, \mathbf{y}_i^t)$ and $h(\mathbf{v}^*, \mathbf{v}^t)$ as a distance from \mathbf{v}^t to a KKT point $\mathbf{v}^* = (\mathbf{x}_j^*, \mathbf{y}_i^*)$:

$$h(\mathbf{v}^*, \mathbf{v}^t) = \frac{K}{J} \sum_{i=1}^I \frac{1}{2\tau_i \rho} \|\mathbf{y}_i^* - \mathbf{y}_i^{t-1}\|_2^2 + \tilde{\mathcal{L}}_\rho(\mathbf{x}^t, \mathbf{y}^t) + \frac{\rho}{2} \|\mathbf{z}^* - \mathbf{z}^t\|_{\mathbf{Q}}^2 + \boldsymbol{\eta}^T B_\phi(\mathbf{x}^*, \mathbf{x}^t). \quad (89)$$

The following Lemma shows that $h(\mathbf{v}^*, \mathbf{v}^t) \geq 0$.

Lemma 7 Let $h(\mathbf{v}^*, \mathbf{v}^t)$ be defined in (89). Setting $\nu_i = 1 - \frac{1}{\tilde{K}_i}$ and $\tau_i = \frac{K}{\tilde{K}_i(2J-K)}$, we have

$$h(\mathbf{v}^*, \mathbf{v}^t) \geq \frac{\rho}{2} \sum_{i=1}^I \zeta_i \|\mathbf{A}_i^r \mathbf{x}^t - \mathbf{a}_i\|_2^2 + \frac{\rho}{2} \|\mathbf{z}^* - \mathbf{z}^t\|_{\mathbf{Q}}^2 + \sum_{j=1}^J \eta_j B_{\phi_j}(\mathbf{x}_j^*, \mathbf{x}_j^t) \geq 0. \quad (90)$$

where $\zeta_i = \frac{J-K}{\tilde{K}_i(2J-K)} + \frac{1}{d_i} - \frac{K}{J\tilde{K}_i} \geq 0$. Moreover, if $h(\mathbf{v}^*, \mathbf{v}^t) = 0$, then $\mathbf{A}_i^r \mathbf{x}^t = \mathbf{a}_i$, $\mathbf{z}^t = \mathbf{z}^*$ and $B_{\phi_j}(\mathbf{x}_j^*, \mathbf{x}_j^t) = 0$. Thus, (81)-(82) are satisfied.

Proof: Using the convexity of f and (81), we have

$$f(\mathbf{x}^*) - f(\mathbf{x}^t) \leq -\langle \mathbf{A}^T \mathbf{y}^*, \mathbf{x}^* - \mathbf{x}^t \rangle = \sum_{i=1}^I \langle \mathbf{y}_i^*, \mathbf{A}_i^r \mathbf{x}^t - \mathbf{a}_i \rangle. \quad (91)$$

Thus,

$$\begin{aligned} \tilde{\mathcal{L}}_{\rho}(\mathbf{x}^t, \mathbf{y}^t) &= f(\mathbf{x}^t) - f(\mathbf{x}^*) + \sum_{i=1}^I \left\{ \langle \mathbf{y}_i^t, \mathbf{A}_i^r \mathbf{x}^t - \mathbf{a}_i \rangle + \frac{(\gamma_i - \tau_i)\rho}{2} \|\mathbf{A}_i^r \mathbf{x}^t - \mathbf{a}_i\|_2^2 \right\} \\ &\geq \sum_{i=1}^I \left\{ \langle \mathbf{y}_i^t - \mathbf{y}_i^*, \mathbf{A}_i^r \mathbf{x}^t - \mathbf{a}_i \rangle + \frac{(\gamma_i - \tau_i)\rho}{2} \|\mathbf{A}_i^r \mathbf{x}^t - \mathbf{a}_i\|_2^2 \right\} \\ &= \sum_{i=1}^I \left\{ \langle \mathbf{y}_i^{t-1} - \mathbf{y}_i^*, \mathbf{A}_i^r \mathbf{x}^t - \mathbf{a}_i \rangle + \langle \mathbf{y}_i^t - \mathbf{y}_i^{t-1}, \mathbf{A}_i^r \mathbf{x}^t - \mathbf{a}_i \rangle + \frac{(\gamma_i - \tau_i)\rho}{2} \|\mathbf{A}_i^r \mathbf{x}^t - \mathbf{a}_i\|_2^2 \right\} \\ &\geq \sum_{i=1}^I \left[-\frac{K}{2J\tau_i\rho} \|\mathbf{y}_i^{t-1} - \mathbf{y}_i^*\|_2^2 - \frac{J\tau_i\rho}{2K} \|\mathbf{A}_i^r \mathbf{x}^t - \mathbf{a}_i\|_2^2 + \frac{(\gamma_i + \tau_i)\rho}{2} \|\mathbf{A}_i^r \mathbf{x}^t - \mathbf{a}_i\|_2^2 \right] \\ &= \sum_{i=1}^I \left[-\frac{K}{2J\tau_i\rho} \|\mathbf{y}_i^{t-1} - \mathbf{y}_i^*\|_2^2 + [\gamma_i + (1 - \frac{J}{K})\tau_i] \frac{\rho}{2} \|\mathbf{A}_i^r \mathbf{x}^t - \mathbf{a}_i\|_2^2 \right]. \end{aligned} \quad (92)$$

$h(\mathbf{v}^*, \mathbf{v}^t)$ is reduced to

$$h(\mathbf{v}^*, \mathbf{v}^t) \geq \frac{\rho}{2} \sum_{i=1}^I [\gamma_i + (1 - \frac{J}{K})\tau_i] \|\mathbf{A}_i^r \mathbf{x}^t - \mathbf{a}_i\|_2^2 + \frac{\rho}{2} \|\mathbf{z}^* - \mathbf{z}^t\|_{\mathbf{Q}}^2 + \boldsymbol{\eta}^T B_{\phi}(\mathbf{x}^*, \mathbf{x}^t). \quad (93)$$

Setting $1 - \nu_i = \frac{1}{\tilde{K}_i}$ and $\tau_i = \frac{K}{\tilde{K}_i(2J-K)}$, we have

$$\begin{aligned} \gamma_i + (1 - \frac{J}{K})\tau_i &= (3 - \frac{2K}{J})(1 - \nu_i) + (1 - \frac{K}{J})\tau_i + \frac{1}{d_i} - \frac{2}{\tilde{K}_i} + (1 - \frac{J}{K})\tau_i \\ &= (1 - \frac{K}{J})\frac{1}{\tilde{K}_i} + (2 - \frac{K}{J} - \frac{J}{K})\frac{K}{\tilde{K}_i(2J-K)} + \frac{1}{d_i} - \frac{K}{J\tilde{K}_i} \\ &= \frac{(J-K)}{\tilde{K}_i(2J-K)} + \frac{1}{d_i} - \frac{K}{J\tilde{K}_i} \geq 0. \end{aligned} \quad (94)$$

Therefore, $h(\mathbf{v}^*, \mathbf{v}^t) \geq 0$. Letting $\zeta_i = \frac{J-K}{\tilde{K}_i(2J-K)} + \frac{1}{d_i} - \frac{K}{J\tilde{K}_i}$ completes the proof. \blacksquare

The following theorem shows that $h(\mathbf{v}^*, \mathbf{v}^t)$ decreases monotonically and thus establishes the global convergence of PDMM.

Theorem 5 (Global Convergence of PDMM) Let $\mathbf{v}^t = (\mathbf{x}_{j_t}^t, \mathbf{y}_i^t)$ be generated by PDMM (5)-(7) and $\mathbf{v}^* = (\mathbf{x}_j^*, \mathbf{y}_i^*)$ be a KKT point satisfying (81)-(82). Setting $\nu_i = 1 - \frac{1}{K_i}$ and $\tau_i = \frac{K}{K_i(2J-K)}$, we have

$$0 \leq \mathbb{E}_{\xi_t} h(\mathbf{v}^*, \mathbf{v}^{t+1}) \leq \mathbb{E}_{\xi_{t-1}} h(\mathbf{v}^*, \mathbf{v}^t), \quad \mathbb{E}_{\xi_t} R(\mathbf{x}^{t+1}) \rightarrow 0. \quad (95)$$

Proof: Adding (91) and (75) yields

$$\begin{aligned} 0 \leq & \sum_{i=1}^I \left\{ \langle \mathbf{y}_i^* - \mathbf{y}_i^t, \mathbf{A}_i^r \mathbf{x}^t - \mathbf{a}_i \rangle + \frac{\tau_i \rho}{2} \|\mathbf{A}_i^r \mathbf{x}^t - \mathbf{a}_i\|_2^2 \right\} \\ & + \frac{J}{K} \left\{ \tilde{\mathcal{L}}_\rho(\mathbf{x}^t, \mathbf{y}^t) - \mathbb{E}_{\mathbb{I}_t} \tilde{\mathcal{L}}_\rho(\mathbf{x}^{t+1}, \mathbf{y}^{t+1}) - \frac{\rho}{2} \sum_{i=1}^I \beta_i \mathbb{E}_{\mathbb{I}_t} \|\mathbf{A}_i^r \mathbf{x}^{t+1} - \mathbf{a}_i\|_2^2 \right. \\ & + \frac{\rho}{2} (\|\mathbf{z}^* - \mathbf{z}^t\|_{\mathbf{Q}}^2 - \mathbb{E}_{\mathbb{I}_t} \|\mathbf{z}^* - \mathbf{z}^{t+1}\|_{\mathbf{Q}}^2 - \mathbb{E}_{\mathbb{I}_t} \|\mathbf{z}^{t+1} - \mathbf{z}^t\|_{\mathbf{P}_t}^2) \\ & \left. + \boldsymbol{\eta}^T (B_\phi(\mathbf{x}^*, \mathbf{x}^t) - \mathbb{E}_{\mathbb{I}_t} B_\phi(\mathbf{x}^*, \mathbf{x}^{t+1}) - \mathbb{E}_{\mathbb{I}_t} B_\phi(\mathbf{x}^{t+1}, \mathbf{x}^t)) \right\}. \end{aligned} \quad (96)$$

Using (6), we have

$$\begin{aligned} \langle \mathbf{y}_i^* - \mathbf{y}_i^t, \mathbf{A}_i^r \mathbf{x}^t - \mathbf{a}_i \rangle + \frac{\tau_i \rho}{2} \|\mathbf{A}_i^r \mathbf{x}^t - \mathbf{a}_i\|_2^2 &= \frac{1}{\tau_i \rho} \langle \mathbf{y}_i^* - \mathbf{y}_i^t, \mathbf{y}_i^t - \mathbf{y}_i^{t-1} \rangle + \frac{\tau_i \rho}{2} \|\mathbf{A}_i^r \mathbf{x}^t - \mathbf{a}_i\|_2^2 \\ &= \frac{1}{2\tau_i \rho} (\|\mathbf{y}_i^* - \mathbf{y}_i^{t-1}\|_2^2 - \|\mathbf{y}_i^* - \mathbf{y}_i^t\|_2^2). \end{aligned} \quad (97)$$

Plugging back into (96) gives

$$\begin{aligned} 0 \leq & \sum_{i=1}^I \frac{1}{2\tau_i \rho} (\|\mathbf{y}_i^* - \mathbf{y}_i^{t-1}\|_2^2 - \|\mathbf{y}_i^* - \mathbf{y}_i^t\|_2^2) \\ & + \frac{J}{K} \left\{ \tilde{\mathcal{L}}_\rho(\mathbf{x}^t, \mathbf{y}^t) - \mathbb{E}_{\mathbb{I}_t} \tilde{\mathcal{L}}_\rho(\mathbf{x}^{t+1}, \mathbf{y}^{t+1}) - \frac{\rho}{2} \sum_{i=1}^I \beta_i \mathbb{E}_{\mathbb{I}_t} \|\mathbf{A}_i^r \mathbf{x}^{t+1} - \mathbf{a}_i\|_2^2 \right. \\ & + \frac{\rho}{2} (\|\mathbf{z}^* - \mathbf{z}^t\|_{\mathbf{Q}}^2 - \mathbb{E}_{\mathbb{I}_t} \|\mathbf{z}^* - \mathbf{z}^{t+1}\|_{\mathbf{Q}}^2 - \mathbb{E}_{\mathbb{I}_t} \|\mathbf{z}^{t+1} - \mathbf{z}^t\|_{\mathbf{P}_t}^2) \\ & \left. + \boldsymbol{\eta}^T (B_\phi(\mathbf{x}^*, \mathbf{x}^t) - \mathbb{E}_{\mathbb{I}_t} B_\phi(\mathbf{x}^*, \mathbf{x}^{t+1}) - \mathbb{E}_{\mathbb{I}_t} B_\phi(\mathbf{x}^{t+1}, \mathbf{x}^t)) \right\} \\ & = \frac{J}{K} \{ h(\mathbf{v}^*, \mathbf{v}^t) - \mathbb{E}_{\mathbb{I}_t} h(\mathbf{v}^*, \mathbf{v}^{t+1}) - \mathbb{E}_{\mathbb{I}_t} R(\mathbf{x}^{t+1}) \}. \end{aligned} \quad (98)$$

Taking expectation over ξ_{t-1} , we have

$$0 \leq \frac{J}{K} \{ \mathbb{E}_{\xi_{t-1}} h(\mathbf{v}^*, \mathbf{v}^t) - \mathbb{E}_{\xi_t} h(\mathbf{v}^*, \mathbf{v}^{t+1}) - \mathbb{E}_{\xi_t} R(\mathbf{x}^{t+1}) \}. \quad (99)$$

Since $\mathbb{E}_{\xi_t} R(\mathbf{x}^{t+1}) \geq 0$, we have

$$\mathbb{E}_{\xi_t} h(\mathbf{v}^*, \mathbf{v}^{t+1}) \leq \mathbb{E}_{\xi_{t-1}} h(\mathbf{v}^*, \mathbf{v}^t). \quad (100)$$

Thus, $\mathbb{E}_{\xi_t} h(\mathbf{v}^*, \mathbf{v}^{t+1})$ converges monotonically.

Rearranging the terms in (99) yields

$$\mathbb{E}_{\xi_t} R(\mathbf{x}^{t+1}) \leq \mathbb{E}_{\xi_{t-1}} h(\mathbf{v}^*, \mathbf{v}^t) - \mathbb{E}_{\xi_t} h(\mathbf{v}^*, \mathbf{v}^{t+1}). \quad (101)$$

Summing over t gives

$$\sum_{t=0}^{T-1} \mathbb{E}_{\xi_t} R(\mathbf{x}^{t+1}) \leq h(\mathbf{v}^*, \mathbf{v}^0) - \mathbb{E}_{\xi_{T-1}} h(\mathbf{v}^*, \mathbf{v}^T) \leq h(\mathbf{v}^*, \mathbf{v}^0). \quad (102)$$

where the last inequality uses the Lemma 7. As $T \rightarrow \infty$, $\mathbb{E}_{\xi_t} R(\mathbf{x}^{t+1}) \rightarrow 0$, which completes the proof. \blacksquare

The following theorem establishes the iteration complexity of PDMM in an ergodic sense.

Theorem 6 *Let $(\mathbf{x}_j^t, \mathbf{y}_i^t)$ be generated by PDMM (5)-(7). Let $\bar{\mathbf{x}}^T = \sum_{t=1}^T \mathbf{x}^t$. Setting $\nu_i = 1 - \frac{1}{K_i}$ and $\tau_i = \frac{K}{K_i(2J-K)}$, we have*

$$\mathbb{E} f(\bar{\mathbf{x}}^T) - f(\mathbf{x}^*) \leq \frac{\sum_{i=1}^I \frac{1}{2\tau_i\rho} \|\mathbf{y}_i^0\|_2^2 + \frac{J}{K} \left\{ \frac{1}{2\beta_i\rho} \|\mathbf{y}_i^*\|_2^2 + \tilde{\mathcal{L}}_\rho(\mathbf{x}^1, \mathbf{y}^1) + \frac{\rho}{2} \|\mathbf{z}^* - \mathbf{z}^1\|_{\mathbf{Q}}^2 + \eta^T B_\phi(\mathbf{x}^*, \mathbf{x}^1) \right\}}{T}, \quad (103)$$

$$\mathbb{E} \sum_{i=1}^I \beta_i \|\mathbf{A}_i^r \bar{\mathbf{x}}^T - \mathbf{a}_i\|_2^2 \leq \frac{2}{\rho} h(\mathbf{v}^*, \mathbf{v}^0). \quad (104)$$

where $\beta_i = \frac{K}{JK_i}$.

Proof: Using (7), we have

$$\begin{aligned} & - \sum_{i=1}^I \left\{ \langle \mathbf{y}_i^t, \mathbf{A}_i^r \mathbf{x}^t - \mathbf{a}_i \rangle - \frac{\tau_i \rho}{2} \|\mathbf{A}_i^r \mathbf{x}^t - \mathbf{a}_i\|_2^2 \right\} \\ &= - \sum_{i=1}^I \left\{ \frac{1}{\tau_i \rho} \langle \mathbf{y}_i^t, \mathbf{y}_i^t - \mathbf{y}_i^{t-1} \rangle - \frac{1}{2\tau_i \rho} \|\mathbf{y}_i^t - \mathbf{y}_i^{t-1}\|_2^2 \right\} \\ &= \sum_{i=1}^I \frac{1}{2\tau_i \rho} (\|\mathbf{y}_i^{t-1}\|_2^2 - \|\mathbf{y}_i^t\|_2^2). \end{aligned} \quad (105)$$

Plugging back into (75) yields

$$\begin{aligned} f(\mathbf{x}^t) - f(\mathbf{x}^*) &\leq \sum_{i=1}^I \frac{1}{2\tau_i \rho} (\|\mathbf{y}_i^{t-1}\|_2^2 - \|\mathbf{y}_i^t\|_2^2) \\ &+ \frac{J}{K} \left\{ \tilde{\mathcal{L}}_\rho(\mathbf{x}^t, \mathbf{y}^t) - \mathbb{E}_{\mathbb{I}_t} \tilde{\mathcal{L}}_\rho(\mathbf{x}^{t+1}, \mathbf{y}^{t+1}) - \frac{\rho}{2} \sum_{i=1}^I \beta_i \mathbb{E}_{\mathbb{I}_t} \|\mathbf{A}_i^r \mathbf{x}^{t+1} - \mathbf{a}_i\|_2^2 \right. \\ &+ \frac{\rho}{2} (\|\mathbf{z}^* - \mathbf{z}^t\|_{\mathbf{Q}}^2 - \mathbb{E}_{\mathbb{I}_t} \|\mathbf{z}^* - \mathbf{z}^{t+1}\|_{\mathbf{Q}}^2 - \mathbb{E}_{\mathbb{I}_t} \|\mathbf{z}^{t+1} - \mathbf{z}^t\|_{\mathbf{P}_t}^2) \\ &\left. + \eta^T (B_\phi(\mathbf{x}^*, \mathbf{x}^t) - \mathbb{E}_{\mathbb{I}_t} B_\phi(\mathbf{x}^*, \mathbf{x}^{t+1}) - \mathbb{E}_{\mathbb{I}_t} B_\phi(\mathbf{x}^{t+1}, \mathbf{x}^t)) \right\}. \end{aligned} \quad (106)$$

Taking expectation over ξ_{t-1} , we have

$$\begin{aligned}
\mathbb{E}_{\xi_{t-1}} f(\mathbf{x}^t) - f(\mathbf{x}^*) &\leq \sum_{i=1}^I \frac{1}{2\tau_i\rho} (\mathbb{E}_{\xi_{t-2}} \|\mathbf{y}_i^{t-1}\|_2^2 - \mathbb{E}_{\xi_{t-1}} \|\mathbf{y}_i^t\|_2^2) \\
&+ \frac{J}{K} \left\{ \mathbb{E}_{\xi_{t-1}} \tilde{\mathcal{L}}_\rho(\mathbf{x}^t, \mathbf{y}^t) - \mathbb{E}_{\xi_t} \tilde{\mathcal{L}}_\rho(\mathbf{x}^{t+1}, \mathbf{y}^{t+1}) - \frac{\rho}{2} \sum_{i=1}^I \beta_i \mathbb{E}_{\xi_t} \|\mathbf{A}_i^r \mathbf{x}^{t+1} - \mathbf{a}_i\|_2^2 \right. \\
&+ \frac{\rho}{2} (\mathbb{E}_{\xi_{t-1}} \|\mathbf{z}^* - \mathbf{z}^t\|_{\mathbf{Q}}^2 - \mathbb{E}_{\xi_t} \|\mathbf{z}^* - \mathbf{z}^{t+1}\|_{\mathbf{Q}}^2 - \mathbb{E}_{\xi_t} \|\mathbf{z}^{t+1} - \mathbf{z}^t\|_{\mathbf{P}_t}^2) \\
&\left. + \boldsymbol{\eta}^T (\mathbb{E}_{\xi_{t-1}} B_\phi(\mathbf{x}^*, \mathbf{x}^t) - \mathbb{E}_{\xi_t} B_\phi(\mathbf{x}^*, \mathbf{x}^{t+1}) - \mathbb{E}_{\xi_t} B_\phi(\mathbf{x}^{t+1}, \mathbf{x}^t)) \right\}. \tag{107}
\end{aligned}$$

Summing over t , we have

$$\begin{aligned}
\sum_{t=1}^T \mathbb{E}_{\xi_{t-1}} f(\mathbf{x}^t) - f(\mathbf{x}^*) &\leq \sum_{i=1}^I \frac{1}{2\tau_i\rho} (\|\mathbf{y}_i^0\|_2^2 - \mathbb{E}_{\xi_{T-1}} \|\mathbf{y}_i^T\|_2^2) \\
&+ \frac{J}{K} \left\{ \tilde{\mathcal{L}}_\rho(\mathbf{x}^1, \mathbf{y}^1) - \mathbb{E}_{\xi_T} \tilde{\mathcal{L}}_\rho(\mathbf{x}^{T+1}, \mathbf{y}^{T+1}) - \frac{\rho}{2} \sum_{t=1}^T \sum_{i=1}^I \beta_i \mathbb{E}_{\xi_t} \|\mathbf{A}_i^r \mathbf{x}^{t+1} - \mathbf{a}_i\|_2^2 \right. \\
&+ \frac{\rho}{2} (\|\mathbf{z}^* - \mathbf{z}^1\|_{\mathbf{Q}}^2 - \mathbb{E}_{\xi_T} \|\mathbf{z}^* - \mathbf{z}^{T+1}\|_{\mathbf{Q}}^2 - \mathbb{E}_{\xi_T} \|\mathbf{z}^{T+1} - \mathbf{z}^T\|_{\mathbf{Q}}^2) \\
&\left. + \boldsymbol{\eta}^T (B_\phi(\mathbf{x}^*, \mathbf{x}^1) - \mathbb{E}_{\xi_T} B_\phi(\mathbf{x}^*, \mathbf{x}^{T+1}) - \mathbb{E}_{\xi_T} B_\phi(\mathbf{x}^{T+1}, \mathbf{x}^T)) \right\}. \tag{108}
\end{aligned}$$

Using (91), we have

$$\begin{aligned}
\tilde{\mathcal{L}}_\rho(\mathbf{x}^{T+1}, \mathbf{y}^{T+1}) &= f(\mathbf{x}^{T+1}) - f(\mathbf{x}^*) + \sum_{i=1}^I [\langle \mathbf{y}_i^{T+1}, \mathbf{A}_i \mathbf{x}^{T+1} - \mathbf{a}_i \rangle + \frac{(\gamma_i - \tau_i)\rho}{2} \|\mathbf{A}_i \mathbf{x}^{T+1} - \mathbf{a}_i\|_2^2] \\
&\geq - \sum_{i=1}^I \langle \mathbf{y}_i^*, \mathbf{A}_i^r \mathbf{x}^{T+1} - \mathbf{a}_i \rangle + \sum_{i=1}^I [\langle \mathbf{y}_i^T, \mathbf{A}_i \mathbf{x}^{T+1} - \mathbf{a}_i \rangle + \frac{(\gamma_i + \tau_i)\rho}{2} \|\mathbf{A}_i \mathbf{x}^{T+1} - \mathbf{a}_i\|_2^2] \\
&\geq - \sum_{i=1}^I \left(\frac{1}{2\delta_i} \|\mathbf{y}_i^*\|_2^2 + \frac{\delta_i}{2} \|\mathbf{A}_i^r \mathbf{x}^{T+1} - \mathbf{a}_i\|_2^2 \right) + \sum_{i=1}^I \left[-\frac{K}{2J\tau_i\rho} \|\mathbf{y}_i^T\|_2^2 + [\gamma_i + (1 - \frac{J}{K})\tau_i] \frac{\rho}{2} \|\mathbf{A}_i \mathbf{x}^{T+1} - \mathbf{a}_i\|_2^2 \right] \\
&\geq - \sum_{i=1}^I \left(\frac{1}{2\delta_i} \|\mathbf{y}_i^*\|_2^2 + \frac{\delta_i}{2} \|\mathbf{A}_i^r \mathbf{x}^{T+1} - \mathbf{a}_i\|_2^2 \right) - \sum_{i=1}^I \frac{K}{2J\tau_i\rho} \|\mathbf{y}_i^T\|_2^2, \tag{109}
\end{aligned}$$

where $\delta_i > 0$ and the last inequality uses (94). Plugging into (108), we have

$$\begin{aligned}
\sum_{t=1}^T \mathbb{E}_{\xi_{t-1}} f(\mathbf{x}^t) - f(\mathbf{x}^*) &\leq \sum_{i=1}^I \frac{1}{2\tau_i\rho} \|\mathbf{y}_i^0\|_2^2 + \frac{J}{K} \left\{ \tilde{\mathcal{L}}_\rho(\mathbf{x}^1, \mathbf{y}^1) + \frac{\rho}{2} \|\mathbf{z}^* - \mathbf{z}^1\|_{\mathbf{Q}}^2 + \boldsymbol{\eta}^T B_\phi(\mathbf{x}^*, \mathbf{x}^1) \right\} \\
&+ \frac{J}{K} \left\{ \sum_{i=1}^I \left[\frac{1}{2\delta_i} \|\mathbf{y}_i^*\|_2^2 + \frac{\delta_i - \beta_i\rho}{2} \mathbb{E} \|\mathbf{A}_i^r \mathbf{x}^{T+1} - \mathbf{a}_i\|_2^2 \right] \right\}. \tag{110}
\end{aligned}$$

Settin $\delta_i = \beta_i\rho$, dividing by T and letting $\bar{\mathbf{x}}^T = \frac{1}{T} \sum_{t=1}^T \mathbf{x}^t$ complete the proof.

Dividing both sides of (102) by T yields (104). ■

B Connection to ADMM

We use ADMM to solve (1), similar as [38, 26] but with different forms. We show that ADMM is a special case of PDMM. The connection can help us understand why the two parameters τ_i, ν_i in PDMM are necessary. We first introduce splitting variables \mathbf{z}_i as follows:

$$\min \sum_{j=1}^J f_j(\mathbf{x}_j) \quad \text{s.t.} \quad \mathbf{A}_j \mathbf{x}_j = \mathbf{z}_j, \sum_{j=1}^J \mathbf{z}_j = \mathbf{a}, \quad (111)$$

which can be written as

$$\min \sum_{j=1}^K f_j(\mathbf{x}_j) + g(\mathbf{z}) \quad \text{s.t.} \quad \mathbf{A}_j \mathbf{x}_j = \mathbf{z}_j, \quad (112)$$

where $g(\mathbf{z})$ is an indicator function of $\sum_{j=1}^K \mathbf{z}_j = \mathbf{a}$. The augmented Lagrangian is

$$\mathcal{L}_\rho(\mathbf{x}_j, \mathbf{z}_j, \mathbf{y}_j) = \sum_{j=1}^J \left[f_j(\mathbf{x}_j) + \langle \mathbf{y}_j, \mathbf{A}_j \mathbf{x}_j - \mathbf{z}_j \rangle + \frac{\rho}{2} \|\mathbf{A}_j \mathbf{x}_j - \mathbf{z}_j\|_2^2 \right], \quad (113)$$

where \mathbf{y}_j is the dual variable. We have the following ADMM iterates:

$$\mathbf{x}_j^{t+1} = \operatorname{argmin}_{\mathbf{x}_i} f_j(\mathbf{x}_j) + \langle \mathbf{y}_j^t, \mathbf{A}_j \mathbf{x}_j - \mathbf{z}_j^t \rangle + \frac{\rho}{2} \|\mathbf{A}_j \mathbf{x}_j - \mathbf{z}_j^t\|_2^2, \quad (114)$$

$$\mathbf{z}^{t+1} = \operatorname{argmin}_{\sum_{j=1}^K \mathbf{z}_j = \mathbf{a}} \sum_{j=1}^K \left[\langle \mathbf{y}_j^t, \mathbf{A}_j \mathbf{x}_j^{t+1} - \mathbf{z}_j \rangle + \frac{\rho}{2} \|\mathbf{A}_j \mathbf{x}_j^{t+1} - \mathbf{z}_j\|_2^2 \right], \quad (115)$$

$$\mathbf{y}_j^{t+1} = \mathbf{y}_j^t + \rho(\mathbf{A}_j \mathbf{x}_j^{t+1} - \mathbf{z}_j^{t+1}). \quad (116)$$

The Lagrangian of (115) is

$$\mathcal{L} = \sum_{j=1}^J \left[\langle \mathbf{y}_j^t, \mathbf{A}_j \mathbf{x}_j^{t+1} - \mathbf{z}_j \rangle + \frac{\rho}{2} \|\mathbf{A}_j \mathbf{x}_j^{t+1} - \mathbf{z}_j\|_2^2 \right] + \langle \boldsymbol{\lambda}, \sum_{j=1}^J \mathbf{z}_j - \mathbf{a} \rangle, \quad (117)$$

where $\boldsymbol{\lambda}$ is the dual variable. The first order optimality is

$$-\mathbf{y}_j^t + \rho(\mathbf{z}_j^{t+1} - \mathbf{A}_j \mathbf{x}_j^{t+1}) + \boldsymbol{\lambda} = 0. \quad (118)$$

Using (116) gives

$$\boldsymbol{\lambda} = \mathbf{y}_j^{t+1}, \quad \forall j. \quad (119)$$

Denoting $\mathbf{y}^t = \mathbf{y}_j^t$, (118) becomes

$$\mathbf{y}^{t+1} = \mathbf{y}^t + \rho(\mathbf{A}_j \mathbf{x}_j^{t+1} - \mathbf{z}_j^{t+1}). \quad (120)$$

Summing over j and using the constraint $\sum_{j=1}^J \mathbf{z}_j = \mathbf{a}$, we have

$$\mathbf{y}^{t+1} = \mathbf{y}^t + \frac{\rho}{J}(\mathbf{A} \mathbf{x}^{t+1} - \mathbf{a}). \quad (121)$$

Subtracting (120) from (121), simple calculations yields

$$\mathbf{z}_j^{t+1} = \mathbf{A}_j \mathbf{x}_j^{t+1} + \frac{1}{J}(\mathbf{A} \mathbf{x}^{t+1} - \mathbf{a}). \quad (122)$$

Plugging back int (114), we have

$$\begin{aligned} \mathbf{x}_j^{t+1} &= \operatorname{argmin}_{\mathbf{x}_j} f_j(\mathbf{x}_j) + \langle \mathbf{y}^t, \mathbf{A}_j \mathbf{x}_j \rangle + \frac{\rho}{2} \|\mathbf{A}_j \mathbf{x}_j - \mathbf{z}_j^t\|_2^2 \\ &= \operatorname{argmin}_{\mathbf{x}_j} f_j(\mathbf{x}_j) + \langle \mathbf{y}^t, \mathbf{A}_j \mathbf{x}_j \rangle + \frac{\rho}{2} \|\mathbf{A}_j \mathbf{x}_j - \mathbf{A}_j \mathbf{x}_j^t + \frac{\mathbf{A} \mathbf{x}^t - \mathbf{a}}{J}\|_2^2 \\ &= \operatorname{argmin}_{\mathbf{x}_j} f_j(\mathbf{x}_j) + \langle \hat{\mathbf{y}}^t, \mathbf{A}_j \mathbf{x}_j \rangle + \frac{\rho}{2} \|\mathbf{A}_j \mathbf{x}_j + \sum_{k \neq j} \mathbf{A}_k \mathbf{x}_k^t - \mathbf{a}\|_2^2, \end{aligned} \quad (123)$$

where $\hat{\mathbf{y}}^t = \mathbf{y}^t - (1 - \frac{1}{J})\rho(\mathbf{A} \mathbf{x}^t - \mathbf{a})$, which becomes PDMM by setting $\tau = \frac{1}{J}, \nu = 1 - \frac{1}{J}$ and updating all blocks. Therefore, sADMM is a special case of PDMM.

C Connection to PJADMM

We consider the case when all blocks are used in PDMM. We show that if setting η_j sufficiently large, the dual backward step (7) is not needed, which becomes PJADMM [9].

Corollary 1 *Let $\{\mathbf{x}_j^t, \mathbf{y}_i^t\}$ be generated by PDMM (5)-(7). Assume $\tau_i > 0$ and $\nu_i \geq 0$. We have*

$$\begin{aligned} f(\mathbf{x}^{t+1}) - f(\mathbf{x}^*) &\leq \sum_{i=1}^I \left\{ -\langle \mathbf{y}_i^{t+1}, \mathbf{A}_i^r \mathbf{x}^{t+1} - \mathbf{a}_i \rangle + \frac{\tau_i \rho}{2} \|\mathbf{A}_i^r \mathbf{x}^{t+1} - \mathbf{a}_i\|_2^2 \right\} \\ &+ \frac{\rho}{2} (\|\mathbf{z}^t - \mathbf{z}^*\|_{\mathbf{Q}}^2 - \|\mathbf{z}^{t+1} - \mathbf{z}^*\|_{\mathbf{Q}}^2 - \|\mathbf{z}^{t+1} - \mathbf{z}^t\|_{\mathbf{Q}}^2) \\ &+ \frac{\rho}{2} \sum_{i=1}^I \left\{ (\nu_i - 1 + \frac{1}{d_i}) (\|\mathbf{A}_i^r \mathbf{x}^t - \mathbf{a}_i\|_2^2 - \|\mathbf{A}_i^r \mathbf{x}^{t+1} - \mathbf{a}_i\|_2^2) \right. \\ &\left. + (\tau_i + 2\nu_i - 2) \|\mathbf{A}_i^r \mathbf{x}^{t+1} - \mathbf{a}_i\|_2^2 + (1 - \nu_i - \frac{1}{d_i}) \|\mathbf{A}_i^r (\mathbf{x}^{t+1} - \mathbf{x}^t)\|_2^2 \right\} \\ &+ \sum_{j=1}^J \eta_j \left(B_{\phi_j}(\mathbf{x}_j^*, \mathbf{x}_j^t) - B_{\phi_j}(\mathbf{x}_j^*, \mathbf{x}_j^{t+1}) - B_{\phi_j}(\mathbf{x}_j^{t+1}, \mathbf{x}_j^t) \right). \end{aligned} \quad (124)$$

Proof: Let \mathbb{I}_t be all blocks, $K = J$. According the definition of \mathbf{P}_t in (42) and \mathbf{Q} in (48), $\mathbf{P}_t = \mathbf{Q}$. Therefore, (51) reduces to

$$\begin{aligned}
f(\mathbf{x}^{t+1}) - f(\mathbf{x}^*) &\leq \sum_{i=1}^I \left\{ -\langle \mathbf{y}_i^{t+1}, \mathbf{A}_i^r \mathbf{x}^{t+1} - \mathbf{a}_i \rangle + \frac{\tau_i \rho}{2} \|\mathbf{A}_i^r \mathbf{x}^{t+1} - \mathbf{a}_i\|_2^2 \right\} \\
&+ \frac{\rho}{2} (\|\mathbf{z}^t - \mathbf{z}^*\|_{\mathbf{Q}}^2 - \|\mathbf{z}^{t+1} - \mathbf{z}^*\|_{\mathbf{Q}}^2 - \|\mathbf{z}^{t+1} - \mathbf{z}^t\|_{\mathbf{Q}}^2) \\
&+ \sum_{j=1}^J \eta_j \left(B_{\phi_j}(\mathbf{x}_j^*, \mathbf{x}_j^t) - B_{\phi_j}(\mathbf{x}_j^*, \mathbf{x}_j^{t+1}) - B_{\phi_j}(\mathbf{x}_j^{t+1}, \mathbf{x}_j^t) \right) \\
&+ \frac{\rho}{2} \sum_{i=1}^I \left\{ (\nu_i - 1 + \frac{1}{d_i}) \|\mathbf{A}_i^r \mathbf{x}^t - \mathbf{a}_i\|_2^2 - (1 - \nu_i - \tau_i + \frac{1}{d_i}) \|\mathbf{A}_i^r \mathbf{x}^{t+1} - \mathbf{a}_i\|_2^2 + (1 - \nu_i - \frac{1}{d_i}) \|\mathbf{A}_i^r (\mathbf{x}^{t+1} - \mathbf{x}^t)\|_2^2 \right\}.
\end{aligned} \tag{125}$$

Rearranging the terms completes the proof. \blacksquare

Corollary 2 Let $\{\mathbf{x}_j^t, \mathbf{y}_i^t\}$ be generated by PDMM (5)-(7). Assume (1) $\tau_i > 0$ and $\nu_i \geq 0$; (2) $\eta_j > 0$; (3) ϕ_j is α_j -strongly convex. We have

$$\begin{aligned}
f(\mathbf{x}^{t+1}) - f(\mathbf{x}^*) &\leq \sum_{i=1}^I \left\{ -\langle \mathbf{y}_i^{t+1}, \mathbf{A}_i^r \mathbf{x}^{t+1} - \mathbf{a}_i \rangle + \frac{\tau_i \rho}{2} \|\mathbf{A}_i^r \mathbf{x}^{t+1} - \mathbf{a}_i\|_2^2 \right\} \\
&+ \frac{\rho}{2} (\|\mathbf{z}^t - \mathbf{z}^*\|_{\mathbf{Q}}^2 - \|\mathbf{z}^{t+1} - \mathbf{z}^*\|_{\mathbf{Q}}^2 - \|\mathbf{z}^{t+1} - \mathbf{z}^t\|_{\mathbf{Q}}^2) \\
&+ \sum_{j=1}^J \eta_j \left(B_{\phi_j}(\mathbf{x}_j^*, \mathbf{x}_j^t) - B_{\phi_j}(\mathbf{x}_j^*, \mathbf{x}_j^{t+1}) \right).
\end{aligned} \tag{126}$$

ν_i and τ_i satisfy $\nu_i \in [1 - \frac{1}{d_i} - \frac{\eta_j \alpha_j}{\rho I d_i \lambda_{\max}^{ij}}, 1 - \frac{1}{d_i}]$ and $\tau_i \leq 1 + \frac{1}{d_i} - \nu_i$, where λ_{\max}^{ij} is the largest eigenvalue of $\mathbf{A}_{ij}^T \mathbf{A}_{ij}$. In particular, if $\eta_j = \frac{(d_i - 1) \rho I \lambda_{\max}^{ij}}{\alpha_j}$, $\nu_i = 0$ and $\tau_i \leq 1 + \frac{1}{d_i}$.

Proof: Assume $\eta_j > 0$. We can choose larger τ_i and smaller ν_i than Lemma 5 by setting η_j sufficiently large. Since ϕ_j is α_j -strongly convex, $B_{\phi_j}(\mathbf{x}_j^{t+1}, \mathbf{x}_j^t) \geq \frac{\alpha_j}{2} \|\mathbf{x}_j^{t+1} - \mathbf{x}_j^t\|_2^2$. We have

$$\sum_{j=1}^J \eta_j B_{\phi_j}(\mathbf{x}_j^{t+1}, \mathbf{x}_j^t) \geq \sum_{i=1}^I \sum_{j=1}^J \frac{\eta_j \alpha_j}{2I} \|\mathbf{x}_j^{t+1} - \mathbf{x}_j^t\|_2^2 \geq \sum_{i=1}^I \sum_{j \in \mathcal{N}(i)} \frac{\eta_j \alpha_j}{2I \lambda_{\max}^{ij}} \|\mathbf{A}_{ij}(\mathbf{x}_j^{t+1} - \mathbf{x}_j^t)\|_2^2. \tag{127}$$

$$\|\mathbf{A}_i^r (\mathbf{x}^{t+1} - \mathbf{x}^t)\|_2^2 = \left\| \sum_{j \in \mathcal{N}(i)} \mathbf{A}_{ij}(\mathbf{x}_j^{t+1} - \mathbf{x}_j^t) \right\|_2^2 \leq d_i \sum_{j \in \mathcal{N}(i)} \|\mathbf{A}_{ij}(\mathbf{x}_j^{t+1} - \mathbf{x}_j^t)\|_2^2, \tag{128}$$

where λ_{\max}^{ij} is the largest eigenvalue of $\mathbf{A}_{ij}^T \mathbf{A}_{ij}$. Plugging into (124) gives

$$\begin{aligned}
f(\mathbf{x}^{t+1}) - f(\mathbf{x}^*) &\leq \sum_{i=1}^I \left\{ -\langle \mathbf{y}_i^{t+1}, \mathbf{A}_i^r \mathbf{x}^{t+1} - \mathbf{a}_i \rangle + \frac{\tau_i \rho}{2} \|\mathbf{A}_i^r \mathbf{x}^{t+1} - \mathbf{a}_i\|_2^2 \right\} \\
&+ \frac{\rho}{2} (\|\mathbf{z}^t - \mathbf{z}^*\|_{\mathbf{Q}}^2 - \|\mathbf{z}^{t+1} - \mathbf{z}^*\|_{\mathbf{Q}}^2 - \|\mathbf{z}^{t+1} - \mathbf{z}^t\|_{\mathbf{Q}}^2) \\
&+ \frac{\rho}{2} \sum_{i=1}^I \left\{ \left(\nu_i - 1 + \frac{1}{d_i} \right) (\|\mathbf{A}_i^r \mathbf{x}^t - \mathbf{a}_i\|_2^2 - \|\mathbf{A}_i^r \mathbf{x}^{t+1} - \mathbf{a}_i\|_2^2) \right. \\
&+ \left. (\tau_i + 2\nu_i - 2) \|\mathbf{A}_i^r \mathbf{x}^{t+1} - \mathbf{a}_i\|_2^2 + \sum_{j \in \mathcal{N}(i)} \left[(1 - \nu_i) d_i - 1 - \frac{\eta_j \alpha_j}{\rho I \lambda_{\max}^{ij}} \right] \|\mathbf{A}_{ij} (\mathbf{x}_j^{t+1} - \mathbf{x}_j^t)\|_2^2 \right\} \\
&+ \sum_{j=1}^J \eta_j \left(B_{\phi_j}(\mathbf{x}_j^*, \mathbf{x}_j^t) - B_{\phi_j}(\mathbf{x}_j^*, \mathbf{x}_j^{t+1}) \right). \tag{129}
\end{aligned}$$

If $(1 - \nu_i) d_i - 1 - \frac{\eta_j \alpha_j}{\rho I \lambda_{\max}^{ij}} \leq 0$, i.e., $\nu_i \geq 1 - \frac{1}{d_i} - \frac{\eta_j \alpha_j}{\rho I d_i \lambda_{\max}^{ij}}$, we have

$$\begin{aligned}
f(\mathbf{x}^{t+1}) - f(\mathbf{x}^*) &\leq \frac{\rho}{2} \sum_{i=1}^I \left\{ -\frac{2}{\rho} \langle \mathbf{y}_i^{t+1}, \mathbf{A}_i^r \mathbf{x}^{t+1} - \mathbf{a}_i \rangle + \tau_i \|\mathbf{A}_i^r \mathbf{x}^{t+1} - \mathbf{a}_i\|_2^2 \right\} \\
&+ \frac{\rho}{2} (\|\mathbf{z}^t - \mathbf{z}^*\|_{\mathbf{Q}}^2 - \|\mathbf{z}^{t+1} - \mathbf{z}^*\|_{\mathbf{Q}}^2 - \|\mathbf{z}^{t+1} - \mathbf{z}^t\|_{\mathbf{Q}}^2) \\
&+ \sum_{i=1}^I \eta_i \left(B_{\phi_i}(\mathbf{x}_i^*, \mathbf{x}_i^t) - B_{\phi_i}(\mathbf{x}_i^*, \mathbf{x}_i^{t+1}) \right) \\
&+ \frac{\rho}{2} \sum_{i=1}^I \left\{ -\left(\nu_i - 1 + \frac{1}{d_i} \right) \|\mathbf{A}_i^r \mathbf{x}^{t+1} - \mathbf{a}_i\|_2^2 + (\tau_i - 2 + 2\nu_i) \|\mathbf{A}_i^r \mathbf{x}^{t+1} - \mathbf{a}_i\|_2^2 \right\}. \tag{130}
\end{aligned}$$

If $\tau_i - 2 + 2\nu_i - (\nu_i - 1 + \frac{1}{d_i}) \leq 0$, i.e., $\tau_i \leq 1 + \frac{1}{d_i} - \nu_i$, the last two terms in (130) can be removed. Therefore, when $\nu_i \geq 1 - \frac{1}{d_i} - \frac{\eta_j \alpha_j}{\rho I d_i \lambda_{\max}^{ij}}$ and $\tau_i \leq 1 + \frac{1}{d_i} - \nu_i$, we have (126). \blacksquare

Define the current iterate $\mathbf{v}^t = (\mathbf{x}_j^t, \mathbf{y}_i^t)$ and $h(\mathbf{v}^*, \mathbf{v}^t)$ as a distance from \mathbf{v}^t to a KKT point $\mathbf{v}^* = (\mathbf{x}_j^*, \mathbf{y}_i^*)$:

$$h(\mathbf{v}^*, \mathbf{v}^t) = \sum_{i=1}^I \frac{1}{2\tau_i \rho} \|\mathbf{y}_i^* - \mathbf{y}_i^t\|_2^2 + \frac{\rho}{2} \|\mathbf{u}^t - \mathbf{u}^*\|_{\mathbf{Q}}^2 + \sum_{j=1}^J \eta_j B_{\phi_j}(\mathbf{x}_j^*, \mathbf{x}_j^t). \tag{131}$$

The following theorem shows that $h(\mathbf{v}^*, \mathbf{v}^t)$ decreases monotonically and thus establishes the global convergence of PDMM.

Theorem 7 (Global Convergence of PDMM) *Let $\mathbf{v}^t = (\mathbf{x}_j^t, \mathbf{y}_i^t)$ be generated by PDMM (5)-(7) and $\mathbf{v}^* = (\mathbf{x}_j^*, \mathbf{y}_i^*)$ be a KKT point satisfying (81)-(82). Assume τ_i, ν_i and γ_i satisfy conditions in Lemma 2. Then \mathbf{v}^t converges to the KKT point \mathbf{v}^* monotonically, i.e.,*

$$h(\mathbf{v}^*, \mathbf{v}^{t+1}) \leq h(\mathbf{v}^*, \mathbf{v}^t) \tag{132}$$

Proof: Adding (91) and (126) together yields

$$\begin{aligned}
0 &\leq \sum_{i=1}^I \left\{ \langle \mathbf{y}_i^* - \mathbf{y}_i^{t+1}, \mathbf{A}_i^r \mathbf{x}^{t+1} - \mathbf{a}_i \rangle + \frac{\tau_i \rho}{2} \|\mathbf{A}_i^r \mathbf{x}^{t+1} - \mathbf{a}_i\|_2^2 \right\} \\
&+ \frac{\rho}{2} (\|\mathbf{u}^t - \mathbf{u}^*\|_{\mathbf{Q}}^2 - \|\mathbf{u}^{t+1} - \mathbf{u}^*\|_{\mathbf{Q}}^2 - \|\mathbf{u}^{t+1} - \mathbf{u}^t\|_{\mathbf{Q}}^2) \\
&+ \sum_{j=1}^J \eta_j \left(B_{\phi_j}(\mathbf{x}_j^*, \mathbf{x}_j^t) - B_{\phi_j}(\mathbf{x}_j^*, \mathbf{x}_j^{t+1}) \right). \tag{133}
\end{aligned}$$

The first term in the bracket can be rewritten as

$$\begin{aligned}
\langle \mathbf{y}_i^* - \mathbf{y}_i^{t+1}, \mathbf{A}_i^r \mathbf{x}^{t+1} - \mathbf{a}_i \rangle &= \frac{1}{\tau_i \rho} \langle \mathbf{y}_i^* - \mathbf{y}_i^{t+1}, \mathbf{y}_i^{t+1} - \mathbf{y}_i^t \rangle \\
&= \frac{1}{2\tau_i \rho} (\|\mathbf{y}_i^* - \mathbf{y}_i^t\|_2^2 - \|\mathbf{y}_i^* - \mathbf{y}_i^{t+1}\|_2^2 - \|\mathbf{y}_i^{t+1} - \mathbf{y}_i^t\|_2^2) \\
&= \frac{1}{2\tau_i \rho} (\|\mathbf{y}_i^* - \mathbf{y}_i^t\|_2^2 - \|\mathbf{y}_i^* - \mathbf{y}_i^{t+1}\|_2^2) - \frac{\tau_i \rho}{2} \|\mathbf{A}_i^r \mathbf{x}^{t+1} - \mathbf{a}_i\|_2^2. \tag{134}
\end{aligned}$$

Plugging back into (133) yields

$$\begin{aligned}
0 &\leq \sum_{i=1}^I \frac{1}{2\tau_i \rho} (\|\mathbf{y}_i^* - \mathbf{y}_i^t\|_2^2 - \|\mathbf{y}_i^* - \mathbf{y}_i^{t+1}\|_2^2) \\
&+ \frac{\rho}{2} (\|\mathbf{u}^t - \mathbf{u}^*\|_{\mathbf{Q}}^2 - \|\mathbf{u}^{t+1} - \mathbf{u}^*\|_{\mathbf{Q}}^2 - \|\mathbf{u}^{t+1} - \mathbf{u}^t\|_{\mathbf{Q}}^2) \\
&+ \sum_{j=1}^J \eta_j \left(B_{\phi_j}(\mathbf{x}_j^*, \mathbf{x}_j^t) - B_{\phi_j}(\mathbf{x}_j^*, \mathbf{x}_j^{t+1}) \right). \tag{135}
\end{aligned}$$

Rearranging the terms completes the proof. ■

The following theorem establishes the $O(1/T)$ convergence rate for the objective in an ergodic sense.

Theorem 8 *Let $(\mathbf{x}_j^t, \mathbf{y}_i^t)$ be generated by PDMM (5)-(7). Assume $\tau_i, \nu_i \geq 0$ satisfy conditions in Lemma 2. Let $\bar{\mathbf{x}}^T = \sum_{t=1}^T \mathbf{x}^t$. We have*

$$f(\bar{\mathbf{x}}^T) - f(\mathbf{x}^*) \leq \frac{\frac{1}{2\tau\rho} \|\mathbf{y}^0\|_2^2 + \frac{\rho}{2} \|\mathbf{u}^0 - \mathbf{u}^*\|_{\mathbf{Q}}^2 + \sum_{j=1}^J \eta_j B_{\phi_j}(\mathbf{x}_j^*, \mathbf{x}_j^0)}{T}, \tag{136}$$

Proof: Using (6), we have

$$\begin{aligned}
-\langle \mathbf{y}_i^{t+1}, \mathbf{A}_i^r \mathbf{x}^{t+1} - \mathbf{a}_i \rangle &= -\frac{1}{\tau_i \rho} \langle \mathbf{y}_i^{t+1}, \mathbf{y}_i^{t+1} - \mathbf{y}_i^t \rangle \\
&= \frac{1}{2\tau_i \rho} (\|\mathbf{y}_i^t\|_2^2 - \|\mathbf{y}_i^{t+1}\|_2^2 - \|\mathbf{y}_i^{t+1} - \mathbf{y}_i^t\|_2^2) \\
&= \frac{1}{2\tau_i \rho} (\|\mathbf{y}_i^t\|_2^2 - \|\mathbf{y}_i^{t+1}\|_2^2) - \frac{\tau_i \rho}{2} \|\mathbf{A}_i^r \mathbf{x}^{t+1} - \mathbf{a}_i\|_2^2. \tag{137}
\end{aligned}$$

Plugging into (126) yields

$$\begin{aligned}
f(\mathbf{x}^{t+1}) - f(\mathbf{x}^*) &\leq \sum_{i=1}^I \frac{1}{2\tau_i\rho} (\|\mathbf{y}_i^t\|_2^2 - \|\mathbf{y}_i^{t+1}\|_2^2) \\
&+ \frac{\rho}{2} (\|\mathbf{u}^t - \mathbf{u}^*\|_{\mathbf{Q}}^2 - \|\mathbf{u}^{t+1} - \mathbf{u}^*\|_{\mathbf{Q}}^2 - \|\mathbf{u}^{t+1} - \mathbf{u}^t\|_{\mathbf{Q}}^2) \\
&+ \sum_{j=1}^J \eta_j \left(B_{\phi_j}(\mathbf{x}_j^*, \mathbf{x}_j^t) - B_{\phi_j}(\mathbf{x}_j^*, \mathbf{x}_j^{t+1}) \right). \tag{138}
\end{aligned}$$

Summing over t from 0 to $T - 1$, we have

$$\begin{aligned}
\sum_{t=0}^{T-1} [f(\mathbf{x}^{t+1}) - f(\mathbf{x}^*)] &\leq \sum_{i=1}^I \frac{1}{2\tau_i\rho} (\|\mathbf{y}_i^0\|_2^2 - \|\mathbf{y}_i^T\|_2^2) \\
&+ \frac{\rho}{2} (\|\mathbf{u}^0 - \mathbf{u}^*\|_{\mathbf{Q}}^2 - \|\mathbf{u}^T - \mathbf{u}^*\|_{\mathbf{Q}}^2) \\
&+ \sum_{j=1}^J \eta_j \left(B_{\phi_j}(\mathbf{x}_j^*, \mathbf{x}_j^0) - B_{\phi_j}(\mathbf{x}_j^*, \mathbf{x}_j^T) \right). \tag{139}
\end{aligned}$$

Applying the Jensen's inequality on the LHS and using $\bar{\mathbf{x}}^T = \sum_{t=1}^T \mathbf{x}^t$ complete the proof. \blacksquare

If $\eta_j = \frac{(d_i-1)\rho L \lambda_{\max}^{ij}}{\alpha_j}$, $\nu_i = 0$ and $\tau_i = 1$. Therefore, PDMM becomes PJADMM [9], where the convergence rate of PJADMM has been improved to $o(1/T)$.